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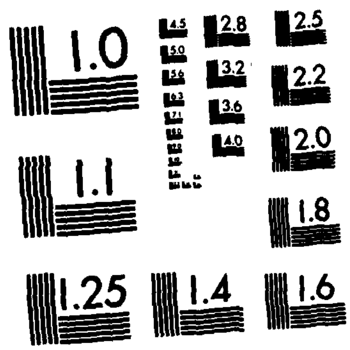

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A TRIDENT SCHOLAR PROJECT REPORT

NO. 117

SEMI-BOOLEAN ALGEBRAS EMPIRICAL LOGIC AND RINGS

UNITED STATES NAVAL ACADEMY
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SEMI-BOOLEAN ALGEBRAS
EMPIRICAL LOGIC
AND
RINGS

A Trident Scholar Project Repott

by

Midshipman Peter P. Haglich, 1982

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Annapolis, Maryland

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Professor James C. Abbott
Mathematics Department

Accepted for Trident Scholar Committee

Chairman

29 July 1982

Date

For Mom, Dad, Helen, and Tony

ABSTRACT

This paper presents applications of semi-Boolean algebras to empirical logic and ring theory. The development of semi-Boolean algebras from subtraction algebras is shown and the identity of the two is established. Examples of subtraction algebras are given. A weakening of one of the subtraction axioms leads to a structure which is non-distributive but orthomodular. Known as orthosubtraction algebra, this structure is identical to a semi-orthomodular lattice. Since the subspaces of a Hilbert space (and thus the projections) form an orthomodular lattice they also form an orthosubtraction algebra. Examples of orthosubtraction algebra applied to Hilbert space are given.

The concept of a manual and how it relates to empirical logic is introduced next. The set of events of a manual is a semi-Boolean algebra. It is atomic and dominated and has relations of operational complementation and operational perspectivity defined on it. From these relations the manual condition is defined and the semi-Boolean algebra is a DASBAM. Examples of manuals and DASBAMs are given. In a DASBAM the operational perspectivity relation is an equivalence relation and a quotient structure of equivalence classes modulo this relation can be formed. Known as the op logic, this structure inherits some properties from the DASBAM. It is not a lattice and it is not distributive, however. It does form what is called an associative orthoalgebra. Examples of op logics are given.

The results from semi-Boolean algebras and DASBAMs can be applied to certain types of rings. Boolean rings form classical DASBAMs. Fields form semi-classical DASBAMs in which every atom is a maximal element and vice versa. Semi-simple rings form DASBAMs which are direct products of field DASBAMs. The projections of rings with involution form associative orthoalgebras from which DASBAMs can be generated. In the special case of a Baer *-ring the projections form an orthomodular lattice.

PREFACE

I would like to thank the Naval Academy and the Trident Scholar Committee for allowing me to undertake this project. It has been the most meaningful experience of my academic career thus far. Not only have I learned a lot about mathematics but I have learned a great deal about mathematical research and research in general. I will benefit from the experience that I have gained in the last year for the rest of my life.

I cannot begin to express my gratitude to my mentor, Professor James C. Abbott, whose enthusiasm and attitude make him the youngest professor at the Naval Academy. I consider myself priveleged to have been accepted as his student and as a member of his family. In the same light I would like to thank his wife, Mrs. Bunny Abbott, for her unique qualities.

For their insight, assistance, and friendship I would like to to thank Assistant Professors Karen Zak and Steven Butcher. I would aldo like to express my appreciation for my fellow Trident Scholar and comrade, Timothy Thomas for his good nature and hard work. Finally I would like to thank Mrs. Ann Hardy for her skillful typing and deciphering of my near-illegible handwriting.

Peter P. Haglich

U.S. Naval Academy
3 May 1982

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INTRODUCTION

This paper presents a summary of the findings and results of my Trident Project, undertaken in the academic year 1981-82. The original working title of this project was "Applications of Orthosubtraction Algebra to Hilbert Space," but as the research progressed the main topic became the application of semi-Boolean algebra to empirical logic. A chapter on orthosubtraction algebra and its application to Hilbert space has been included in the text.

According to Foulis and Randall of the University of Massachusetts the goal of an empirical science is to "order, explain, and predict the observable events associated with certain physical situations or experiments."[1] Empirical logic is the attempt to formalize a logical calculus of experimental propositions for use in the empirical sciences. The level at which this is done is the operational level, hence the term "operational statistics".

In this approach the basic concept is that of an operation. Each operation is a classical experiment with outcomes that form a classical sample space. Techniques of conventional probability theory based on measures on a Boolean algebra can be applied to each operation alone. Often, however, the experimenter has a collection of operations which overlap. The performance of one interferes with the performance of another and a non-classical situation is generated. It is this case that the theory of manuals was formulated to handle.

Foulis and Randall define a manual mathematically as a non-empty set of non-empty sets which are irredundant and in which the manual condition is satisfied.[2] From the operations, subsets of outcomes can be formed. These are known as events since they correspond to events in the classical probability sense. The set of all events forms a semi- Boolean structure as defined by Abbott and his students at the Naval Academy. It is the application of the theory of semi-Boolean algebras to the theory of manuals that is the main focus of this paper.

One such application came from an unexpected area, that of ring theory. Some of the nicest results come from the theory of manuals as applied to certain types of rings. This use of ideas motivated from empirical science in the study of the almost purely abstract area of rings is somewhat serendipitous. It illustrates some of the beauty of mathematics in that it connects seemingly unrelated areas.

In Chapter I the idea of a subtraction algebra is introduced. Subtraction algebras are duals of implication algebras as developed by Abbott and Kleindorfer at the Naval Academy in 1961.[3] In Part One the subtraction axioms are introduced and some properties are derived from them. Theorems 15 and 16 establish the fact that subtraction algebras and semi-Boolean algebras are identical. As a result many of the ideas from semi-Boolean algebras may be applied using subtraction notation. Some examples of subtraction algebras are presented. In Part Two concepts from universal algebra are discussed in the context of subtraction algebras. These include the ideas of subalgebras, homomorphisms, ideals, congruence relations, and diredt products.

Chapter II examines the effects of a weakened form of the third subtraction axiom on the structure. The result is orthosubtraction algebra, which is the dual of the orthoimplication algebra of Abbott and Kimble.[4] Again properties are derived from the axioms. Theorems 14 and 15 establish the correspondence between orthosubtraction algebras and semi-orthomodular lattices. In Part Two these results are studied in the context of Hilbert space. Specifically the application of orthosubtraction to the lattice of subspaces of a Hilbert space and to the set of projection operators on a Hilbert space is made. Some examples of orthosubtraction algebra as applied to particular Hilbert spaces are given.

Chapter III introduces the idea of a manual. Part One discusses the intuitive notion of a manual and presents the set theoretic definitions of Foulis and Randall. Part Two illustrates how subtraction algebra can be applied to manuals to generate DASBAMs. "DASBAM" is an acronym that I coined to stand for Dominated Atomic Semi-Boolean Algebra satisfying the Manual condition. Theorem 3 states the identification between event structures of manuals and DASBAMs. Part Three gives some classifications of DASBAMs. These include classical DASBAMs, semi-classical DASBAMs, dacifications, direct products, and free DASBAMs. The idea of ghosting is defined. Part Four presents some examples of DASBAMs and manuals.

Chapter IV investigates some properties of the operational perspectivity relation defined in the previous chapter. A new structure is formed, known as the op logic. Part One presents some properties of the op logic. The idea of an associative orthoalgebra is introduced and Theorem 13 shows that the op logic of any DASBAM is an associative orthoalgebra. Part Two illustrates some examples of op logics.

Chapter V introduces the subject of ring DASBAMs. Different types of rings are shown to exhibit a partial order. Some of these form DASBAMs directly, others form associative algebras from which DASBAMs can be generated. Part One shows how a Boolean ring with identity forms a Boolean algebra and thus a classical DASBAM. Part Two shows how a field produces a semi-classical DASBAM in which each atom is a maximal element and vice versa. Part Three uses Theorem III-6 to show how finite semi-simple rings form DASBAMs which are direct products of field DASBAMs. In Part Four rings with involution are discussed. They are shown to yield associative orthoalgebras which in turn yield DASBAMs. The special case of a Baer *-ring, which produces an orthomodular lattice is presented.

These subjects are only a small part of the areas that I would have liked to have covered. The application of ideas from probability theory such as weights and states and the notions of property lattices and questions from Piron and Aerts are subjects that I did some study in. Unfortunately I need to study these in more detail in order to present them properly. I hope to have an opportunity to do this and to use this paper as a basis for further research.

CHAPTER I: SUBTRACTION ALGEBRAS

PART ONE: Definitions and Properties

A subtraction algebra $S = (S, \setminus)$ consists of a carrier set S with a binary subtraction operation satisfying three axioms.

<u>S1</u> $x \setminus (y \setminus x) = x$ (contraction)

<u>S2</u> $x \setminus (x \setminus y) = y \setminus (y \setminus x)$ (quasi-commutative)

<u>S3</u> $(x \setminus y) \setminus z = (x \setminus z) \setminus y$ (exchange)

<u>Lemma 1</u>: $(x \setminus y) \setminus y = x \setminus y$

<u>Proof</u>: $(x \setminus y) \setminus y = (x \setminus y) \setminus (y \setminus (x \setminus y)) = x \setminus y$

<u>Lemma 2</u>: $x \setminus x = (y \setminus x) \setminus (y \setminus x)$

<u>Proof</u>: $x \setminus x = x \setminus (x \setminus (y \setminus x)) = (y \setminus x) \setminus ((y \setminus x) \setminus x) = (y \setminus x) \setminus (y \setminus x)$

<u>Theorem 1</u>: There exists a constant $o \in S$ such that

(a) $x \setminus x = o$

(b) $x \setminus o = x$

(c) $o \setminus x = o$

<u>Proof</u>:

(a) It must be shown that $x \setminus x$ is independent of x, that is $x \setminus x = y \setminus y$ for all x and y in S.
$$(x \setminus x) = (y \setminus x) \setminus (y \setminus x) = (y \setminus (y \setminus x)) \setminus (y \setminus (y \setminus x))$$
$$= (x \setminus (x \setminus y)) \setminus (x \setminus (x \setminus y)) = (x \setminus y) \setminus (x \setminus y) = y \setminus y$$

(b) $x \setminus o = x \setminus (x \setminus x) = x$

(c) $o \setminus x = (x \setminus x) \setminus x = x \setminus x = o.$

<u>Theorem 2</u>: For all x and y in S the following are true:

(a) $(x \setminus y) \setminus x = o$

(b) $y \setminus (y \setminus (y \setminus x)) = y \setminus x$

(c) $x \setminus (y \setminus (y \setminus x)) = x \setminus y$

(d) $(y \setminus (y \setminus x)) \setminus x = (y \setminus (y \setminus x)) \setminus y = o$

(e) $(x \setminus y) \setminus (y \setminus x) = x \setminus y$

<u>Proof</u>:

(a) $(x \setminus y) \setminus x = (x \setminus x) \setminus y = o \setminus y = o$

(b) $y \setminus (y \setminus (y \setminus x)) = (y \setminus x) \setminus ((y \setminus x) \setminus y) = (y \setminus x) \setminus o = y \setminus x$

(c) $x \setminus (y \setminus (y \setminus x)) = x \setminus (x \setminus (x \setminus y)) = x \setminus y$

(d) $(y \setminus (y \setminus x)) \setminus x = (x \setminus (x \setminus y)) \setminus x = o$

(e) $(x \setminus y) \setminus (y \setminus x) = (x \setminus (y \setminus x)) \setminus y = x \setminus y$

From these results another Theorem can be established.

__Theorem 3__:

(a) $x \setminus y = y \setminus x$ if and only if $x = y$

(b) $x \setminus y = x$ if and only if $y \setminus x = y$

(c) $x \setminus y = y$ implies $y = o = x$

__Proof__:

(a) Suppose $x \setminus y = y \setminus x$. Then $x = x \setminus (y \setminus x) = x \setminus (x \setminus y) = y \setminus (y \setminus x)$
$= y \setminus (x \setminus y) = y$.

(b) Suppose $x \setminus y = x$. Then $y \setminus x = y \setminus (x \setminus y) = y$.

(c) Suppose $x \setminus y = y$. Then $y = y \setminus (x \setminus y) = y \setminus y = o$

Also $x \setminus y = x \setminus o = x$ and $x = y$ hence $x = o$

From the results in Lemmas 1 and a and Theorems 1 and 2 it is possible to find all elements generated by two general elements x and y. This is known as the free subtraction algebra on two elements and consists of the six element set $\{o, x, y, x \setminus y, y \setminus x, x \setminus (x \setminus y)\}$. The subtraction table is given below:

	o	x	y	$x \setminus y$	$y \setminus x$	$x \setminus (x \setminus y)$
o	o	o	o	o	o	o
x	x	o	$x \setminus y$	$x \setminus (x \setminus y)$	x	$x \setminus y$
y	y	$y \setminus x$	o	y	$x \setminus (x \setminus y)$	$y \setminus x$
$x \setminus y$	$x \setminus y$	o	$x \setminus y$	o	$x \setminus y$	$x \setminus y$
$y \setminus x$	$y \setminus x$	$y \setminus x$	o	$y \setminus x$	o	$y \setminus x$
$x \setminus (x \setminus y)$	$x \setminus (x \setminus y)$	o	o	$x \setminus (x \setminus y)$	$x \setminus (x \setminus y)$	o

Table 1

Theorem 3 allows the construction of special algebras satisfying additional conditions.

The next theorem provides some identities on three elements.

__Theorem 4__: For all x, y, and z in S:

(a) $(z \setminus y) \setminus (z \setminus x) = (x \setminus y) \setminus (x \setminus z)$ __S5__

(b) $z \setminus (z \setminus (y \setminus x)) = (z \setminus x) \setminus (z \setminus (y \setminus x))$

(c) $(y \setminus (y \setminus (z \setminus x))) \setminus (y \setminus x) = o$

(d) $y \setminus z = o$ implies $y \setminus (y \setminus (z \setminus x)) = y \setminus x$

(e) $(z \setminus y) \setminus x = (z \setminus x) \setminus (y \setminus x)$ (autodistributive) __S4__

(f) $z \setminus (z \setminus (y \setminus x)) = (z \setminus (z \setminus y)) \setminus x$

(g) $z \setminus (z \setminus (y \setminus (y \setminus x))) = x \setminus (x \setminus (y \setminus (y \setminus x)))$

<u>Proof</u>:

(a) $(z \setminus y) \setminus (z \setminus x) = (z \setminus (z \setminus x)) \setminus y = (x \setminus (x \setminus z)) \setminus y = (x \setminus y) \setminus (x \setminus z)$

(b) $z \setminus (z \setminus (y \setminus x)) = (y \setminus x) \setminus ((y \setminus x) \setminus z) = ((y \setminus x) \setminus x) \setminus ((y \setminus x) \setminus z)$
$= (z \setminus x) \setminus (z \setminus (y \setminus x))$

(c) $(y \setminus (y \setminus (z \setminus x))) \setminus (y \setminus x) = ((y \setminus x) \setminus (y \setminus (z \setminus x))) \setminus (y \setminus x) = o$

(d) Suppose $y \setminus z = o$. $(y \setminus x) \setminus (y \setminus (y \setminus (z \setminus x))) = (y \setminus (y \setminus (y \setminus (z \setminus x)))) \setminus x$
$= (y \setminus (z \setminus x)) \setminus x = (y \setminus x) \setminus (z \setminus x) = ((y \setminus x) \setminus o) \setminus (z \setminus x) = ((y \setminus x) \setminus (y \setminus z)) \setminus (z \setminus x)$
$= ((z \setminus x) \setminus (z \setminus y)) \setminus (z \setminus x) = o = (y \setminus (y \setminus (z \setminus x))) \setminus (y \setminus x)$
Thus $(y \setminus x) = (y \setminus (y \setminus (z \setminus x)))$ by Theorem 3a.

(e) $((z \setminus y) \setminus x) \setminus ((z \setminus x) \setminus (y \setminus x)) = ((z \setminus x) \setminus y) \setminus ((z \setminus x) \setminus (y \setminus x))$
$= ((y \setminus x) \setminus y) \setminus ((y \setminus x) \setminus (z \setminus x)) = o \setminus ((y \setminus x) \setminus (z \setminus x)) = o$
$((z \setminus x) \setminus (y \setminus x)) \setminus ((z \setminus y) \setminus x) = ((z \setminus x) \setminus (y \setminus x)) \setminus ((z \setminus x) \setminus y)$
$= ((z \setminus x) \setminus ((z \setminus x) \setminus y)) \setminus (y \setminus x) = (y \setminus (y \setminus (z \setminus x))) \setminus (y \setminus x) = o$

Since $((z \setminus y) \setminus x) \setminus ((z \setminus x) \setminus (y \setminus x)) = ((z \setminus x) \setminus (y \setminus x)) \setminus ((z \setminus y) \setminus x)$
then by Theorem 3a $(z \setminus y) \setminus x = (z \setminus x) \setminus (y \setminus x)$

(f) $z \setminus (z \setminus (y \setminus x)) = (y \setminus x) \setminus ((y \setminus x) \setminus z) = (y \setminus x) \setminus ((y \setminus z) \setminus x)$
$= (y \setminus x) \setminus ((y \setminus x) \setminus (z \setminus x)) = (z \setminus x) \setminus ((z \setminus x) \setminus (y \setminus x)) = (z \setminus x) \setminus ((z \setminus y) \setminus x)$
$= (z \setminus (z \setminus y)) \setminus x$ by the autodistributive law.

(g) $z \setminus (z \setminus (y \setminus (y \setminus x))) = (z \setminus (z \setminus y)) \setminus (y \setminus x) = (y \setminus (y \setminus x)) \setminus (y \setminus x)$
$= (x \setminus (y \setminus z)) \setminus (x \setminus y) = (x \setminus (x \setminus y)) \setminus (y \setminus z) = x \setminus (x \setminus (y \setminus (y \setminus z)))$
by using the result of Theorem 4f.

From Theorem 4 alternate characterizations of subtraction algebra can be obtained.

<u>Theorem 5</u>: A subtraction algebra is an algebra satisfying S1 and S5.

<u>Proof</u>: It is necessary to verify S2 and S3.

(S2) $x \setminus (x \setminus y) = (x \setminus (y \setminus x)) \setminus (x \setminus y) = (y \setminus (y \setminus x)) \setminus (y \setminus x) = y \setminus (y \setminus x)$

(S3) It shall be shown that $((z \setminus y) \setminus x) \setminus ((z \setminus x) \setminus y) = o$ and without loss of generality $((z \setminus x) \setminus y) \setminus ((z \setminus y) \setminus x) = o$. Since Theorem 3a was proved only using S1 and S2 the end result will follow.

$((z \setminus y) \setminus x) \setminus ((z \setminus x) \setminus y) = (((z \setminus y) \setminus x) \setminus ((z \setminus y) \setminus z)) \setminus ((z \setminus x) \setminus y)$
$= ((z \setminus x) \setminus (z \setminus (z \setminus y))) \setminus ((z \setminus x) \setminus y) = (y \setminus (z \setminus (z \setminus y))) \setminus (y \setminus (z \setminus x))$
$= (y \setminus (y \setminus (y \setminus z))) \setminus (y \setminus (z \setminus x)) = (y \setminus z) \setminus (y \setminus (z \setminus x)) = ((z \setminus x) \setminus z) \setminus ((z \setminus x) \setminus y)$
$= o \setminus ((z \setminus x) \setminus y) = o$.

Thus $((z \setminus x) \setminus y) \setminus ((z \setminus y) \setminus x) = ((z \setminus y) \setminus x) \setminus ((z \setminus x) \setminus y)$ and hence by Theorem 3a $((z \setminus x) \setminus y) = (z \setminus y) \setminus x)$

From the subtraction identities a partial order relation may be defined on $(S, \setminus)$. A partial order relation is a binary relation which is reflexive, antisymmetric, and transitive.

<u>Definition 1</u>: $x \leq y$ if and only if $x \setminus y = o$

<u>Theorem 6</u>: $(S, \setminus, \leq)$ is a partially ordered set.

<u>Proof</u>: (Reflexivity) $x \leq x$ as $x \setminus x = o$.

(Antisymmetry) Suppose $x \leq y$ and $y \leq x$, that is $x \setminus y = o = y \setminus x$. Then $x = y$ by Theorem 3a.

(Transitivity) Suppose $x \leq y$ and $y \leq z$, that is $x \setminus y = o = y \setminus z$.

$x \setminus z = (x \setminus o) \setminus z = (x \setminus (x \setminus y)) \setminus z = (y \setminus (y \setminus x)) \setminus z = (y \setminus z) \setminus (y \setminus x)$
$= o \setminus (y \setminus x) = 0$, thus $x \leq z$.

<u>Thoerem 7</u>: $x \leq y$ implies $x \setminus z \leq y \setminus z$ (Isotone Law)

<u>Proof</u>: $(x \setminus z) \setminus (y \setminus z) = (x \setminus y) \setminus z = o \setminus z = o$, thus $x \setminus z \leq y \setminus z$.

<u>Theorem 8</u>: $x \leq y$ implies $z \setminus y \leq z \setminus x$ (Antitone Law)

<u>Proof</u>: $(z \setminus y) \setminus (z \setminus x) = (x \setminus y) \setminus (x \setminus z) = o \setminus (x \setminus z) = o$, thus $z \setminus y \leq z \setminus x$.

<u>Theorem 9</u>: o is the greatest lower bound for S.

<u>Proof</u>: It must be shown that o is a lower bound and that if z is any other lower bound for S then $z \leq o$.

First, since $o \setminus x = o$ it follows that $o \leq x$ for all x in S.

Secondly, if z is a lower bound then $z \leq o$, but also $o \leq z$, therefore $o = z$.

<u>Theorem 10</u>: $x \leq y$ if and only if there is some z such that $x = y \setminus z$

<u>Proof</u>: Suppose $x \leq y$, that is $x \setminus y = o$.

$x = x \setminus o = x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

Let $x = y \setminus z$, then $x \setminus y = (y \setminus z) \setminus y = o$, thus $x \leq y$

For each element x in S a subset I(x) of S can be defined, where $I(x) = \{y | y \leq x\}$. Theorem 10 states that $I(x) = \{x \setminus z | z \in S\}$, and is somewhat like a right ideal in other algebraic structures. I(x) is called the principal ideal generated by x.

Given a partially ordered set a diagram may be drawn showing the partial order. The elements are represented by points and the relation $x \leq y$ is represented by a chain of upward segments from x to y. The partial order diagram for the algebra of Table 1 is shown here.

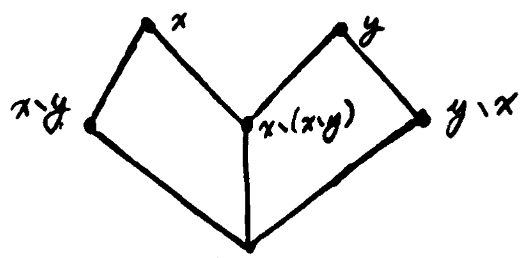


Figure 1

Theorem 2d states that $x\setminus(x\setminus y)$ is a lower bound for x and y, that is $x\setminus(x\setminus y) \leq x$ and $x\setminus(x\setminus y) \leq y$. In general it is desirable to know what the greatest lower bound of x and y is.

<u>Theorem 11</u>: $x\setminus(x\setminus y) = y\setminus(y\setminus x)$ is the greatest lower bound of x and y.

<u>Proof</u>: The above remarks show that $x\setminus(x\setminus y)$ is a lower bound. Suppose z is a lower bound for x and y, that is $z \leq x$, $z \leq y$ or $z\setminus x = o = z\setminus y$.

$z\setminus(x\setminus(x\setminus y)) = (z\setminus(x\setminus(x\setminus y)))\setminus o = (z\setminus(x\setminus(x\setminus y)))\setminus(z\setminus x)$
$= (x\setminus(x\setminus(x\setminus y)))\setminus(x\setminus z) = (x\setminus y)\setminus(x\setminus z) = (z\setminus y)\setminus(z\setminus x) = o\setminus o = o.$
Thus $z \leq x\setminus(x\setminus y)$ and hence $x\setminus(x\setminus y)$ is the greatest lower bound.

This last theorem shows that any pair of elements (and, by induction, any finite set of elements) has a greatest lower bound, or "meet." We will denote this meet by $x \wedge y$ or in case there are more than two elements, $\bigwedge_I x_i$ where I is an index set. However, there is not necessarily an upper bound for any general pair of elements. An example of this is shown in Figure 1, where x and y do not have a least upper bound, or "join" (denoted $x \vee y$). A structure closed under both meets and joins is called a lattice. A structure closed only under meets (or joins) is called a meet (or join) semi-lattice.

<u>Corollary</u>: $(S, \setminus, \leq, \wedge)$ is a meet semi-lattice.

There are some cases where a least upper bound does exist.

<u>Theorem 12</u>: $x \vee y$ exists if and only if there exists $z \in S$ such that $x \leq z$, $y \leq z$. Furthermore $x \vee y = z\setminus((z\setminus x)\setminus y)$.

<u>Proof</u>: Suppose $x \leq z$, $y \leq z$. Firstly, $(z\setminus x)\setminus y = (z\setminus y)\setminus(x\setminus y)$
$= (z\setminus y)\setminus((x\setminus y)\setminus o) = (z\setminus y)\setminus((x\setminus y)\setminus(x\setminus z)) = (z\setminus y)\setminus((z\setminus y)\setminus(z\setminus x)) = (z\setminus y)\wedge(z\setminus x)$
Thus $z\setminus((z\setminus x)\setminus y) = z\setminus((z\setminus x)\wedge(z\setminus y))$.

Next $x\setminus(z\setminus((z\setminus x)\setminus y)) = x\setminus(z\setminus((z\setminus x)\wedge(z\setminus y))) \leq x\setminus(z\setminus(z\setminus x))$ by applying the antitone law twice. $x\setminus(z\setminus(z\setminus x)) = x\setminus(x\setminus(x\setminus z)) = x\setminus z = o = y\setminus(z\setminus((z\setminus x)\setminus y))$ without loss of generality and $z\setminus((z\setminus x)\setminus y)$ is an upper bound for x and y.

Finally, suppose $x \leq p$ and $y \leq p$. It must be shown that $(z\setminus((z\setminus x)\setminus y))\setminus p = o$.

$(z\setminus((z\setminus x)\setminus y))\setminus p = (z\setminus p)\setminus(((z\setminus x)\setminus y)\setminus p)$
$= (z\setminus p)\setminus((z\setminus p)\setminus(y\setminus p)) = (z\setminus p)\setminus((z\setminus p)\setminus o) = (z\setminus p)\setminus(z\setminus p) = o$ and hence $z\setminus((z\setminus x)\setminus y) \leq p$.

Therefore $x \vee y = z\setminus((z\setminus x)\setminus y)$.

<u>Corollary</u>: For all $x \in S$ $I(x)$ is a lattice.

<u>Proof</u>: Let $y \in I(x)$ and $z \in I(x)$, that is $y \leq x$ and $z \leq x$. $y \wedge z \leq y \leq x$ implies $y \wedge z \in I(x)$.

Also, since x is an upper bound $z \vee y$ exists and $z \vee y \leq x$, therefore $z \vee y \in I(x)$.

<u>Theorem 13</u>: (Distributive Laws). If x,y, and z have a common upper bound then

(a) $(x \wedge y) \vee z = (x \vee z)\wedge(y \vee z)$

(b) $(x \vee y)\wedge z = (x \wedge z)\vee(y \wedge z)$

<u>Proof</u>: (b) will be proved, then (a) will follow from (b).

To show (b), note that $z \leq (x \vee z)$ and $z \leq (y \vee z)$, thus $z \leq (x \vee z)\wedge(y \vee z)$. Also $x \wedge y \leq x \leq x \vee z$ and $x \wedge y \leq y \leq y \vee z$, thus $x \wedge y \leq (x \vee z)\wedge(y \vee z)$. Therefore $(x \wedge y)\vee z \leq (x \vee z)\wedge(y \vee z)$.

Let r denote $(x \vee z)\wedge(y \vee z)$ and u denote $x \vee z$. Clearly $r \leq u$ and thus $r\setminus z \leq u\setminus z$

Also $u = u\setminus((u\setminus z)\setminus x)$

thus $u\setminus z = (u\setminus((u\setminus z)\setminus x))\setminus z = (u\setminus z)\setminus((u\setminus z)\setminus x)$
$= (u\setminus z)\setminus((u\setminus x)\setminus z) = (u\setminus(u\setminus x))\setminus z = x\setminus z \leq x$

Since $r\setminus z = x\setminus z$ it follows that $r\setminus z \leq x$

Without loss of generality $r\setminus z \leq y$

Hence $r\setminus z \leq x \wedge y$ and $(r\setminus z)\setminus(x \wedge y) = o$

$r = r\setminus o = r\setminus((r\setminus z)\setminus(x \wedge y)) = z \vee (x \wedge y)$, the desired result.

For the proof of (a) use (b), thus

$(x \vee z)\wedge(y \vee z) = (x \wedge(y \vee z))\vee(z \wedge (y \vee z)) = (x \wedge y)\vee(x \wedge z) \vee z$
$= z \vee (x \wedge y)$.

<u>Corollary</u>: I(x) is a distributive lattice.

An overall complement cannot be defined on S as there is not in general an upper bound. However, a relative complement can be defined.

<u>Theorem 14</u>: If $x \leq y$ then $y\setminus x$ is the complement of x relative to y.

<u>Proof</u>: (a) $x \wedge (y\setminus x) = x\setminus(x\setminus(y\setminus x)) = x\setminus x = o$

(b) $x \vee (y\setminus x) = y\setminus((y\setminus x)\setminus(y\setminus x)) = y\setminus o = y$

(c) $y\setminus(y\setminus x) = y \wedge x = x$, thus $(x'_y)'_y = x$.

<u>Definition 2</u>: A Boolean algebra is a complemented distributive lattice with an upper bound and a lower bound.

<u>Corollary</u>: For all x in S, I(x) is a Boolean algebra.

More general than a Boolean algebra is a semi-Boolean algebra.

<u>Definition 3</u>: A semi-Boolean algebra is a meet semi-lattice in which every principal ideal is a Boolean algebra.

<u>Theorem 15</u>: Any subtraction algebra is a semi-Boolean algebra.

<u>Proof</u>: Follows from the corollaries to Theorem 12 and Theorem 14.

Theorem 16: Any semi-Boolean algebra is a subtraction algebra where subtraction is defined as $x \setminus y = (x \wedge y)'_x$ where $'_x$ denotes the relative complement with respect to x.

Proof: The three axioms must be verified.

(S1) $x \setminus (y \setminus x) = x \setminus (x \wedge y)'_y = (x \wedge (x \wedge y)'_y)'_x$

$= (x \wedge (y \wedge (x \wedge y)'_y))'_x = ((x \wedge y)'_y \wedge (x \wedge y))'_x = o'_x = x$

(S2) $x \setminus (x \setminus y) = x \setminus (x \wedge y)'_x = (x \wedge (x \wedge y)'_x)'_x = ((x \wedge y)'_x)'_x = x \wedge y$

which is symmetric in x and y and thus $x \setminus (x \setminus y) = y \setminus (y \setminus x)$

(S3) First $x'_{z \setminus y} = x'_z \wedge (z \setminus y)$, thus $(z \setminus y) \setminus x = ((z \setminus y) \wedge x)'_{z \setminus y}$

$= ((z \setminus y) \wedge x)'_z \wedge (z \setminus y) = ((z \setminus y) \wedge (x \wedge z))'_z \wedge (z \setminus y) = ((z \wedge y)'_z \wedge (z \wedge x)'_z) \wedge (z$

$= ((z \setminus x) \wedge (z \setminus y)) \vee ((z \setminus y)'_z \wedge (z \setminus y)) = (z \setminus x) \wedge (z \setminus y)$ is symmetric in

x and y, thus $(z \setminus y) \setminus x = (z \setminus x) \setminus y$

Theorems 15 and 16 together yield the result that the categories of semi-Boolean algebras and subtraction algebras are identical.

The corollary to Theorem 14 states that every principal ideal in a subtraction algebra is a Boolean algebra. What if the entire algebra is a principal ideal?

Theorem 17: $(S, \setminus)$ is a Boolean algebra if and only if there exists a constant $1 \in S$ satisfying $x \leq 1$ for all $x \in S$.

Proof: S is the same as I(1) and is thus Boolean.

This definition of Boolean algebra, utilizing one binary operation, one nullary operation, and three axioms is perhaps the simplest possible. By Theorem 5 the simplest axiom set is S1, S5 and Theorem 17.

The following are examples of subtraction algebras:

Example 1: A Boolean algebra $(B, \vee, \wedge, o, 1, ')$ is a subtraction algebra under $x \setminus y = x \wedge y'$.

Example 2: Let X be any set, A and B subsets of X. Define $A \setminus B = \{a \in A | a \notin B\}$. The power set P(X), the set of all subsets of X is a subtraction algebra under this subtraction.

Example 3: Take X as in the previous example and take 10(X) to be the subcollection of P(X) consisting of all subsets of X having 10 elements or less. Define subtraction on 10(X) the same way as on P(X). Since $A \setminus B$ has fewer elements than A if $A \in 10(X)$ then $A \setminus B \in 10(X)$ for all B. The axioms check out and thus 10(X) is a subtraction algebra. It should be noted that if A and B are each 10 element sets and $A \neq B$ then $A \cup B$ has more than 10 elements and thus A and B have no upper bound in 10(X). Therefore this is an example of a subtraction algebra which is not a lattice.

Further analogous examples can be generated by considering the set of all finite subsets of X or the set of all countable subsets of X.

Example 4: Let $(R,+,\circ)$ be a ring satisfying $x^2 = x$ for all x in R. This ring can be shown to be commutative and of characteristic 2 ($x + x = o$ for all $x \in R$). Define $x \setminus y = x + xy$. The axioms are verified:

(S1) $x \setminus (y \setminus x) = x \setminus (y + xy) = x + xy + x^2 y = x + xy + xy = x$

(S2) $x \setminus (x \setminus y) = x \setminus (x + xy) = x + x^2 + x^2 y = x + x + xy = xy = yx = y \setminus (y \setminus x)$
(Thus $x \wedge y = xy$)

(S3) $(z \setminus x) \setminus y = (z + zx) \setminus y = z + zx + zy + zxy$ is symmetric in x and y, thus $(z \setminus x) \setminus y = (z \setminus y) \setminus x$.

Any Boolean ring is a subtraction algebra. Furthermore a Boolean ring with identity becomes a Boolean algebra where $x \wedge y = xy$, $x \vee y = x + y + xy$, and $x' = 1 + x$.

Even without an identity $x \vee y = x + y + xy$ exists for all x and y and therefore R is a lattice, though it is not necessarily bounded above. This is known as a generalized Boolean algebra.

PART TWO: The Algebra of Subtraction Algebra

Since a subtraction algebra is an algebraic structure all of the usual algebraic notions may be applied to the study of subtraction algebras. The first is the concept of a subalgebra.

Definition 4: A subalgebra of a subtraction algebra $(S,\setminus)$ is a non-empty subset $T \subseteq S$ which is closed under subtraction. That is, if $x \in T$ and $y \in T$ then $x \setminus y \in T$.

Lemma 3: If T is a subsubtraction algebra then $o \in T$.

Proof: T is non-empty, thus there exists $x \in T$, T is closed under subtraction, therefore $x \setminus x = o \in T$.

Theorem 18: If $\{T_\alpha\}$ (where $\alpha \in \Gamma$ and Γ is any index set) is a collection of subalgebras then $\bigcap_\Gamma T_\alpha$ is a subsubtraction algebra.

Proof: $o \in T_\alpha$ for all $\alpha \in \Gamma$ implies $o \in \bigcap_\Gamma T_\alpha$, thus $\bigcap_\Gamma T_\alpha$ is non-empty.

Let $x \in \bigcap_\Gamma T_\alpha$ and $y \in \bigcap_\Gamma T_\alpha$, then $x \in T_\alpha$ and $y \in T_\alpha$ for all $\alpha \in \Gamma$, which implies $x \setminus y \in T_\alpha$ for all $\alpha \in \Gamma$ and hence $x \setminus y \in \bigcap_\Gamma T_\alpha$.

This theorem means that one can define the subalgebra generated by a subset $X \subseteq S$ as the intersection of all subalgebras containing X.

Another commonly used idea is that of a homomorphism.

<u>Definition 5</u>: Let S and T be subtraction algebras. A subtraction homomorphism (homomorphism in this context) is a mapping $\phi: S \to T$ which preserves subtraction, that is $\phi(x \setminus y) = \phi(x) \setminus \phi(y)$ for all x and y in S.

<u>Lemma 4</u>: If ϕ is a homomorphism then $\phi(o) = o$

<u>Proof</u>: $\phi(o) = \phi(x \setminus x) = \phi(x) \setminus \phi(x) = o$.

<u>Theorem 19</u>: If ϕ is a homomorphism then ϕ preserves order, that is $x \leq y$ implies $\phi(x) \leq \phi(y)$

<u>Proof</u>: $x \leq y$ implies $x \setminus y = o$, thus $\phi(x) \setminus \phi(y) = \phi(x \setminus y) = \phi(o) = o$ and hence $\phi(x) \leq \phi(y)$

<u>Definition 6</u>: If $\phi: S \to T$ is a homomorphism then the kernel of ϕ is the pre-image of o, or:

$$\ker \phi = \{x \in S \mid \phi(x) = o\}$$

<u>Theorem 20</u>: $\ker \phi$ is a subalgebra of S.

<u>Proof</u>: From Lemma 4 $o \in \ker \phi$ and thus $\ker \phi$ is non-empty.

Let x and y be in the kernel, then $\phi(x) = \phi(y) = o$ $\phi(x \setminus y) = \phi(x) \setminus \phi(y) = o \setminus o = o$ and thus $x \setminus y \in \ker \phi$.

<u>Theorem 21</u>: $x \in \ker \phi$ and $y \leq x$ imply $y \in \ker \phi$

<u>Proof</u>: $x \in \ker \phi$ implies $\phi(x) = o$. $\phi(y) = \phi(y) \setminus o = \phi(y) \setminus \phi(x)$ $= \phi(y \setminus x) = \phi(o) = o$ and thus $y \in \ker \phi$

<u>Theorem 22</u>: $x \in \ker \phi$ and $y \in \ker \phi$ imply $\phi(z \setminus x) = \phi(z \setminus y)$ for all $z \in S$.

<u>Proof</u>: $\phi(z \setminus x) = \phi(z) \setminus \phi(x) = \phi(z) \setminus o = \phi(z) \setminus \phi(y) = \phi(z \setminus y)$

<u>Theorem 23</u>: $x \in \ker \phi$, $y \in \ker \phi$, and $x \vee y$ exists imply $x \vee y \in \ker \phi$.

<u>Proof</u>: Suppose $x \vee y$ exists. Since ϕ preserves order $\phi(x \vee y) = \phi(x) \vee \phi(y) = o \vee o = o$ and therefore $x \vee y \in \ker \phi$.

The previous two results set the stage for the next definition:

<u>Definition 7</u>: An ideal in a subtraction algebra is a subset $I \subseteq S$ satisfying two conditons:

(1) $x \in I$ and $y \leq x$ imply $y \in I$

(2) $x \in I$, $y \in I$, and $x \vee y$ exists imply $x \vee y \in I$ or, equivalently

(1') $x \in I$ and $y \setminus x \in I$ imply $y \in I$

(2') $o \in I$

<u>Theorem 24</u>: If ϕ is a homomorphism then $\ker \phi$ is an ideal.

Proof: Follows from Theorems 21 and 23 and Definition 7.

A closely related idea in algebra is that of a congruence relation.

Definition 8: A congruence relation $\equiv$ is a relation which satisfies the following four properties

(1) $x \equiv x$ (reflexive)

(2) $x \equiv y$ implies $y \equiv x$ (symmetric)

(3) $x \equiv y$ and $y \equiv z$ imply $x \equiv z$ (transitive)

(4) $x \equiv y$ and $w \equiv z$ imply $x \setminus w \equiv y \setminus z$ (substitution property)

The kernel of a congruence relation is the set of all things congruent to o and is designated by ker $\equiv$ or $\hat{o}$.

Theorem 25: If $\equiv$ is a congruence relation then there is a natural homomorphism associated with $\equiv$, $\eta : S \to \hat{S}$ where $\hat{S} = \{\hat{x} | x \in S\}$ and $\hat{x} = \{y | y \equiv x\}$. Furthermore, $\hat{o} = \ker \eta$

Proof: Define $\hat{x} \setminus \hat{y} = \widehat{x \setminus y}$. Since $\equiv$ has the substitution property this subtraction is well defined. From this definition subtraction is obviously preserved and thus η is a homomorphism.

Let $x \in \hat{o}$. $\eta(x) = \hat{x} = \hat{o} = \widehat{x \setminus x} = \hat{x} \setminus \hat{x}$ implies $x \in \ker \hat{\eta}$

Let $x \in \ker \eta$, that is $\eta(x) = \hat{o} = \{y | y \equiv x\}$ implies $x \in \hat{o}$

Since $\ker \eta \leq \hat{o}$ and $\hat{o} \subseteq \ker \eta$ it follows that $\hat{o} = \ker \eta$.

If $I \subseteq S$ is an ideal a congruence relation modulo I can be defined.

Theorem 26: If I is an ideal then the relation $x \equiv y \bmod I$ if and only if $x \setminus y \in I$ and $y \setminus x \in I$ is a congruence relation.

Proof: (1) $o \in I$ implies $x \setminus x \in I$ and thus $x \equiv x \bmod I$.

(2) $x \equiv y \bmod I$ if and only if $y \equiv x \bmod I$ follows from the hypothesis.

(3) Let $x \equiv y \bmod I$ and $y \equiv z \bmod I$. Then $x \setminus y$, $y \setminus x$, $y \setminus z$, and $z \setminus y$ are all in I, which means that $(y \setminus z) \setminus (y \setminus x) \in I$. $(y \setminus z) \setminus (y \setminus x) = (x \setminus z) \setminus (x \setminus y) \in I$. By condition 1' of an ideal $x \setminus z \in I$. Likewise, without loss of generality $z \setminus x \in I$, therefore $x \equiv z \bmod I$.

(4) Let $x \equiv y \bmod I$ and $w \equiv z \bmod I$. First it will be shown that $x \setminus w \equiv y \setminus w \bmod I$, then it will be shown that $y \setminus w \equiv y \setminus z \bmod I$. The end result will follow by transitivity shown above.

(a) $x \setminus y \in I$ and $y \setminus x \in I$ since $x \equiv y \bmod I$.
$(x \setminus w) \setminus (y \setminus w) = (x \setminus y) \setminus w \in I$, also
$(y \setminus w) \setminus (x \setminus w) = (y \setminus x) \setminus w \in I$
Thus $(x \setminus w) \equiv (y \setminus w) \bmod I$

(b) $w \setminus z \in I$ and $z \setminus w \in I$ since $w \equiv z \bmod I$
$(y \setminus w) \setminus (y \setminus z) = (z \setminus w) \setminus (z \setminus y) \in I$, also
$(y \setminus z) \setminus (y \setminus w) = (w \setminus z) \setminus (w \setminus y) \in I$.
Thus $(y \setminus w) \equiv (y \setminus z) \bmod I$.

Therefore $x \setminus w \equiv y \setminus z$ mod I by transitivity.

The machinery is now been established for "The First Fundamental Homomorphism Theorem of Subtraction Algebra."

Theorem 27: Let ϕ be a homomorphism, then the range $\phi(S)$ is isomorphic to the quotient structure $\hat{S}$ modulo ker ϕ.

Proof: It must be shown that there is a one to one, onto mapping from $\phi(S)$ to $\hat{S}$, denoted ψ where $\psi(\phi(x)) = \hat{x}$

(a) (One-to-one). Suppose $\psi(\phi(x)) = \psi(\phi(y))$, then $\hat{x} = \hat{y}$ and hence $x \setminus y \in \ker \phi$ and $y \setminus x \in \ker \phi$. Thus $\phi(x \setminus y) = o = \phi(y \setminus x)$ and $\phi(x) \setminus \phi(y) = o = \phi(y) \setminus \phi(x)$ and therefore $\phi(x) = \phi(y)$

(b) (Onto) Let $\hat{z} \in \hat{S}$. It must be shown that there is a $\phi(w) \in \phi(S)$ such that $\psi(\phi(w)) = \hat{z}$. Choose $w = z$, then $\psi(\phi(z)) = \hat{z}$. Therefore $\phi(S)$ and $\hat{S}$ are isomorphic.

As the preceding results indicate many of the ideas associated with homomorphisms in other systems of algebra can be defined when considering subtraction algebras. Terms such as injections, surjections, isomorphisms, automorphisms, etc. can be defined. One can speak of the automorphism group of a subtraction algebra.

One other algebraic concept needs to be introduced.

Definition 9: The direct or Cartesian product of two subtraction algebras S and T is defined as the set of all ordered pairs (s,t) where $s \in S$ and $t \in T$, that is

$$S \times T = \{(s,t) \mid s \in S \text{ and } t \in T\}.$$

Subtraction is defined $(s,t) \setminus (s',t') = (s \setminus s', t \setminus t')$. It is easy to verify S1 through S3 and thus $S \times T$ is a subtraction algebra. This concept can be extended to arbitrary direct products in the usual way.

Finally the idea of a free algebra can be defined.

Definition 10: The free subtraction algebra generated by a set of elements A, S(A) is the set of all unique elements that can be derived by combining elements in A and making only the identifications derived from S1, S2, and S3.

The algebra given in Table 1 and Figure 1 is the free subtraction algebra on two generators, $S(\{x, y\})$.

This section has shown that most of the ideas of universal algebra apply to subtraction algebra. Specifically, the category of subtraction algebra is closed under subalgebras, homomorphisms, and direct products.

CHAPTER II: ORTHOSUBTRACTION ALGEBRAS

PART ONE: Definitions and Properties

The properties of subtraction algebras given in the previous chapter were derived from the axioms S1, S2, and S3. In this chapter the effects of weakening S3 on the structure will be examined.

<u>Definition 1</u>: An orthosubtraction algebra is a structure $\mathcal{S} = (S, \setminus)$ where S is a carrier set of elements and $\setminus$ is a binary operation satisfying three axioms.

<u>S1</u> $x \setminus (y \setminus x) = x$

<u>S2</u> $x \setminus (x \setminus y) = y \setminus (y \setminus x)$

<u>OS3</u> $(z \setminus (x \setminus y)) \setminus x = z \setminus x$

Axioms S1 and S2 are the contraction and quasicommutative axioms from subtraction algebra. Axiom OS3 is a weakening of the exchange axiom.

<u>Theorem 1</u>: If $\mathcal{S}$ is a subtraction algebra then it is an orthosubtraction algebra.

<u>Proof</u>: It is only necessary to verify OS3. Let $\mathcal{S}$ be a subtraction algebra, then it satisfies the autodistributive law. Hence:

$$(z \setminus (x \setminus y)) \setminus x = (z \setminus x) \setminus ((x \setminus y) \setminus x) = (z \setminus x) \setminus o = z \setminus x$$

The similarity between subtraction and orthosubtraction algebras means that many results from subtraction algebra carry over to orthosubtraction algebra. Specifically, those results derived from S1 and S2 also hold in orthosubtraction algebra and are summarized here without repeating the proofs:

<u>Theorem 2</u>: The following are true in any orthosubtraction algebra:

(a) $(x \setminus y) \setminus y = x \setminus y$

(b) $x \setminus x = (y \setminus x) \setminus (y \setminus x)$

(c) There exists a constant $o \in S$ such that

1. $x \setminus x = o$
2. $x \setminus o = x$
3. $o \setminus x = o$

(d) $x \setminus y = y \setminus x$ if and only if $x = y$

(e) $x \setminus y = y$ if and only if $x = y = o$

<u>Lemma</u>: $(x \setminus y) \setminus x = o$ (OS4)

<u>Proof</u>: $(x \setminus y) \setminus x = ((x \setminus y) \setminus (x \setminus y)) \setminus x = o \setminus x = o$ by applying OS3 and Theorem 2.c.

This result held in subtraction algebra but with a different proof. Most of the results of Theorems 1, 2, and 3 from the previous chapter can now be established and are again stated without proof.

<u>Theorem 3</u>: The following are true in any orthosubtraction algebra:

(a) $y\setminus(y\setminus(y\setminus x)) = y\setminus x$

(b) $x\setminus(y\setminus(y\setminus x)) = x\setminus y$

(c) $(y\setminus(y\setminus x))\setminus x = (y\setminus(y\setminus x))\setminus y = o$

(d) $x\setminus y = x$ if and only if $y\setminus x = y$

Only one more lemma is needed.

<u>Lemma</u>: $(x\setminus y)\setminus(y\setminus x) = x\setminus y$

Again, as with subtraction algebra it is possible to find all elements generated by two general elements x and y. This is the free orthosubtraction algebra on two elements. Since the previous results are identical with those from subtraction algebra the table is the same and is not repeated here.

There are two more results which should be stated here.

<u>Theorem 4</u>: (a) $y\setminus x = o$ implies $(z\setminus y)\setminus x = z\setminus x$ (OS5)

(b) $y\setminus x = o$ implies $(z\setminus x)\setminus(z\setminus y) = o$

<u>Proof</u>: (a) $(z\setminus y)\setminus x = (z\setminus(y\setminus o))\setminus x = (z\setminus(y\setminus(y\setminus x))\setminus x =$
$(z\setminus(x\setminus(x\setminus y)))\setminus x = z\setminus x$ by 3C and OS3.

(b) Suppose $y\setminus x = o$. $(z\setminus x)\setminus(z\setminus y) = ((z\setminus y)\setminus x)\setminus(z\setminus y) = o$

It is now possible to give an alternate set of axioms for orthosubtraction algebra.

<u>Theorem 5</u>: An orthosubtraction algebra is an algebra satisfying S1, S2, OS4, and OS5.

<u>Proof</u>: It is only necessary to show that OS3 can be derived from S1, S2, OS4, and OS5.

From OS4 $(x\setminus y)\setminus x = o$, thus by OS5 $(z\setminus(x\setminus y))\setminus x = z\setminus x$

A partial order in terms of the orthosubtraction may now be defined.

<u>Defininition 2</u>: $x \leq y$ if and only if $x\setminus y = o$

<u>Theorem 6</u>: $(S,\setminus,\leq)$ is a partially ordered set

<u>Proof</u>: (1) $x\setminus x = o$ implies $x \leq x$ (reflexivity)

(2) Let $x \leq y$, $y \leq x$; i.e. $x\setminus y = o = y\setminus x$.
Then $x = y$ by Theorem 2d. (anti-symmetry)

(3) Let $x \leq y$, $y \leq z$, i.e. $x\setminus y = o = y\setminus z$
By OS5 $y\setminus z = o$ implies $x\setminus z = (x\setminus y)\setminus z = o\setminus z = o$,
thus $x \leq z$ (transitivity)

Theorem 4b now becomes the antitone law as in subtraction algebra.

<u>Theorem 7</u>: o is the greatest lower bound for S.

<u>Proof</u>: To show that o is the greatest lower bound two things must be established, that $o \leq x$ for all x in S and that if z is also a lower bound then $z \leq o$ (hence $z = o$).

$o \leq x$ follows from Theorem 2.c.

Let $z \leq x$, then $z \leq o$ and $z \setminus o = o$; thus $z = o$ by Theorem 2.e.

<u>Theorem 8</u>: $x \leq y$ if and only if there exists $z \in S$ such that $x = y \setminus z$.

(a) Let $x \leq y$, that is $x \setminus y = o$. Then
$y \setminus (y \setminus (y \setminus (y \setminus x))) = y \setminus ((y \setminus x) \setminus ((y \setminus x) \setminus y)) = y \setminus ((y \setminus x) \setminus o) = y \setminus (y \setminus x)$
$= x \setminus (x \setminus y) = x \setminus o = x$, i.e. $x = y \setminus z$ where $z = y \setminus (y \setminus (y \setminus x))$.

(b) Suppose $x = y \setminus z$ for some z. $x \setminus y = (y \setminus z) \setminus y = o$ and hence $x \leq y$.

The same notion of principal ideal generated by x, $I(x) = \{y \mid y \leq x\}$ $= \{x \setminus z \mid z \in S\}$ can be defined.

<u>Theorem 9</u>: The greatest lower bound of x and y under $\leq$ is $x \setminus (x \setminus y)$.

<u>Proof</u>: From Theorem 3c. $x \setminus (x \setminus y) \leq x$ and $x \setminus (x \setminus y) \leq y$.
Suppose $z \leq x$ and $z \leq y$, that is $z \setminus x = z \setminus y = o$.

$z = z \setminus o = z \setminus (z \setminus y) = y \setminus (y \setminus z) \leq y \setminus (y \setminus x)$ by applying the antitone law twice.

From this theorem it follows that $(S, \setminus, \leq)$ is closed under meets. The meet of x and y, $x \setminus (x \setminus y)$ shall be designated by $x \wedge y$.

<u>Corollary</u>: $(S, \setminus, \leq, \wedge)$ is a meet semi-lattice.

Lower bounds always exist, but under what conditions does a least upper bound exist?

<u>Theorem 10</u>: $x \vee y$ exists if and only if there exists a $z \in S$ such that $x \leq z$ and $y \leq z$. Furthermore $x \vee y = z \setminus ((z \setminus y) \wedge (z \setminus x))$

<u>Proof</u>: Suppose such a z exists, that is $x \setminus z = o = y \setminus z$. It is necessary to show first that $z \setminus ((z \setminus y) \wedge (z \setminus x))$ is an upper bound.

$x \setminus (z \setminus ((z \setminus x) \wedge (z \setminus y)) \leq x \setminus (z \setminus (z \setminus x))$ by applying the antitone law twice.
$x \setminus (z \setminus (z \setminus x)) = x \setminus (x \setminus (x \setminus z)) = x \setminus z = o$, thus $x \leq z \setminus ((z \setminus x) \wedge (z \setminus y))$. Without loss of generality $y \leq z \setminus ((z \setminus x) \wedge (z \setminus y))$.

Finally, suppose $x \leq p$ and $y \leq p$, then $z \setminus p \leq z \setminus x$ and $z \setminus p \leq z \setminus y$, thus $z \setminus p \leq (z \setminus x) \wedge (z \setminus y)$

Applying the antitone law again $z \setminus ((z \setminus x) \wedge (z \setminus y)) \leq z \setminus (z \setminus p)$ and $z \setminus ((z \setminus x) \wedge (z \setminus y)) \leq z \wedge p \leq p$.

Therefore $x \vee y = z \setminus ((z \setminus x) \wedge (z \setminus y))$

Corollary: $I(x)$ is a lattice for all x in S.

This next theorem is the orthomodular law expressed in terms of orthosubtraction, as will be shown later.

Theorem 11: (OMS)

If $x \leq y \leq z$ then $x = y \setminus (z \setminus x)$

Proof: Suppose $x \leq y \leq z$, then $x = x \wedge z = z \setminus (z \setminus x)$
Since $x \leq y$ then by the antitone law $z \setminus y \leq z \setminus x$ and thus $z \setminus (z \setminus x)$
$= (z \setminus (z \setminus y)) \setminus (z \setminus x)$ by OS5.

$(z \setminus (z \setminus y)) \setminus (z \setminus x) = (z \wedge y) \setminus (z \setminus x) = y \setminus (z \setminus x)$ since $z \wedge y = y$.

Theorem 12: If $x \leq z$ then $z \setminus x$ is a relative complement of x in $I(z)$, denoted $x_z^{\perp}$

Proof: $(z \setminus x) \wedge x = (z \setminus x) \setminus ((z \setminus x) \setminus x) = (z \setminus x) \setminus (z \setminus x) = o$.
$(z \setminus x) \vee x = z \setminus ((z \setminus x) \wedge (z \setminus (z \setminus x))) = z \setminus ((z \setminus x) \wedge (z \wedge x)) = z \setminus ((z \setminus x) \setminus x)$
$= z \setminus o = z$.

Definition 2: An orthomodular lattice is a bounded lattice satisfying the following properties:

OM1: $x \vee x^{\perp} = 1$, $x \wedge x^{\perp} = o$

OM2: $x^{\perp\perp} = x$

OM3: If $x \leq y$ then $y^{\perp} \leq x^{\perp}$

OM4: If $x \leq y$ then $x = y \wedge (x \vee y^{\perp})$ (Orthomodular Meet Identity)

Theorem 13: For all z in S $I(z)$ is an orthomodular lattice.

Proof: (OM1) was verified in Theorem 12.

(OM2) $(x_z^{\perp})_z^{\perp} = z \setminus (x_z^{\perp}) = z \setminus (z \setminus x) = z \wedge x = x$

(OM3) Let $x \leq y \leq z$, then $y_z^{\perp} = z \setminus y \leq z \setminus x = x_z^{\perp}$

(OM4) Let $x \leq y \leq z$. By OMS $x = y \setminus (z \setminus x)$.

Also $y \setminus x \leq y \leq z$ and thus $y \setminus x = y \setminus (z \setminus (y \setminus x))$.

$y \wedge (y_z^{\perp} \vee x) = y \wedge ((z \setminus y) \vee x) = y \wedge (z \setminus ((z \setminus x) \setminus y)) = y \wedge (z \setminus (y \setminus (y \setminus (z \setminus x))))$

$= y \wedge (z \setminus (y \setminus x)) = y \setminus (y \setminus (z \setminus (y \setminus x))) = y \setminus (y \setminus x) = y \wedge x = x$.

Hence the orthomodular law is verified.

Corollary to Theorem 13: If S has a greatest element 1 then S is an orthomodular lattice.

Proof: S is the same as $I(1)$ and is hence an orthomodular lattice.

Definition 3: A semi-orthomodular lattice is a meet semi-lattice in which every principal ideal is an orthomodular lattice and which satisfies the compatibility condition: (C) $y \leq x \leq z$ implies $y^{\perp}_{x} = y^{\perp}_{z} \wedge x$

Theorem 14: $(S, \smallsetminus, \leq, \wedge, {}^{\perp}_{x})$ is a semi-orthomodular lattice.

Proof: Only condition C needs to be verified:
$x \wedge y^{\perp}_{z} = x \smallsetminus (x \smallsetminus (z \smallsetminus y)) = x \smallsetminus y = y^{\perp}_{x}$ by OMS.

Every orthosubtraction (algebra) thus determines a semi-orthomodular lattice.

Theorem 15: Every semi-orthomodular lattice determines an orthosubtraction algebra where the subtraction is defined by $y \smallsetminus x = (y \wedge x)^{\perp}_{y}$.

Proof: The axioms need to be verified.

(S1) $x \smallsetminus (y \smallsetminus x) = x \smallsetminus (y \wedge x)^{\perp}_{y} = (x \wedge (y \wedge x)^{\perp}_{y})^{\perp}_{x} = ((x \wedge y) \wedge (x \wedge y)^{\perp}_{y})^{\perp}_{x}$
$= o^{\perp}_{x} = x.$

(S2) $x \smallsetminus (x \smallsetminus y) = x \smallsetminus (x \wedge y)^{\perp}_{x} = ((x \wedge y)^{\perp}_{x} \wedge x)^{\perp}_{x} = ((x \wedge y)^{\perp}_{x})^{\perp}_{x} = x \wedge y$
$= y \smallsetminus (y \smallsetminus x)$

(OS3) For simplicity denote $z \smallsetminus (x \smallsetminus y)$ by w. $w \leq z$ and thus $w = w \wedge z$.

$(z \smallsetminus (x \smallsetminus y)) \smallsetminus x = w \smallsetminus x = (w \wedge x)^{\perp}_{w} = (w \wedge z \wedge x)^{\perp}_{w} = (w \wedge z \wedge x)^{\perp}_{z} \wedge w$
$= (w^{\perp}_{z} \vee (z \wedge x)^{\perp}_{z}) \wedge w = (z \wedge x)^{\perp}_{z} = z \smallsetminus x$ by the orthomodular law

The category of semi-orthomodular lattices is therefore identical to the category of orthosubtraction algebras.

PART TWO: Applications to Hilbert Space

One of the main reasons for the interest in orthomodular lattices is the work of Birkoff and Von Neumann in 1936. In studying quantum mechanics they discovered that the distributive laws failed and thus the structure was not Boolean. As an alternate structure the orthomodular lattice was adopted. This was because of certain properties satisfied by Hilbert space, a mathematical tool much used in the study of quantum mechanics and as a basis of quantum logic. In this section the connection between orthosubtraction algebra and Hilbert space shall be made.

A Hilbert space is a vector space with an inner product defined on it which is complete. It may be finite dimensional or infinite dimensional. Volumes have been written on Hilbert space and I will not repeat the results here. A good book on Hilbert space is Introduction to Hilbert Space by Berberian.[5]

Of particular concern in the study of Hilbert space are the operators on the space, specifically the projections and self-adjoint operators. Since each operator determines a closed linear subspace (or subspace for short) these are also of interest.

Definition 4: (a) A closed linear subspace M of a Hilbert space H is a subset of vectors which is closed under linear combinations and is closed in the metric on H, that is:

$$x \in M \text{ and } y \in M \text{ imply } \alpha x + \beta y \in M$$

and if $\{x_n\}$ is a sequence in M and $\lim\limits_{n\to\infty} x_n$ exists then $\lim\limits_{n\to\infty} x_n \in M$.

(b) A linear mapping on a Hilbert space H is a mapping $T: H \to H$ satisfying $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all x and y in H and α and β in the field of scalars. T is continuous if $\{Tx_n\}$ converges to Tx whenever x_n converges to x. A continuous linear mapping is called an operator. The adjoint of T, T^*, is the operator satisfying $(Tx|y)=(x|T^*y)$ for all x and y in H. T is said to be self-adjoint if $T = T^*$. T is said to be a projection if it is self-adjoint and $T^2 = T$.

Each projection uniquely determines a subspace of H, the range of that projection. Also, given a subspace there is a unique projection of which it is the range. Thus any results about the subspaces can be extended to the projections and vice versa.

It is now time to define subtraction on $L(H)$, the set of all closed linear subspaces of the Hilbert space H.

Definition 5: If M and N are subspaces of H then $M\setminus N = M\cap(M\cap N)^\perp$

Lemma 1: $M\cap N^\perp \le M\setminus N$

Proof: Note that $M\cap N \le N$ implies $N^\perp \le (M\cap N)^\perp$

Let $x \in M\cap N^\perp$. Thus $x \in M$ and $x \in N^\perp$ and hence $x \in (M\cap N)^\perp$. Therefore $x \in M\cap(M\cap N)^\perp = M\setminus N$ and $M\cap N^\perp \subseteq M\setminus N$.

__Theorem 16:__ $M \setminus N = M \setminus (M \wedge N)$

__Proof:__ $M \setminus N = M \wedge (M \wedge N)^{\perp} = M \wedge (M \wedge (M \wedge N))^{\perp} = M \setminus (M \wedge N)$

__Theorem 17:__ $N \leq M$ implies $M \setminus N = M \wedge N^{\perp}$

__Proof:__ $M \setminus N = M \wedge (M \wedge N)^{\perp} = M \wedge N^{\perp}$

__Definition 6:__ If $N \leq M$ then $N^{\perp}_{M} = M \setminus N = M \wedge N^{\perp}$

Note that this is similar to the compatibility condition, as $N \leq M \leq H$ implies that $N^{\perp}_{m} = M \wedge N^{\perp}_{M} = M \wedge N^{\perp}$ by Theorem 17.

__Theorem 18:__ $(L(H), \setminus)$ satisfies OS1.

__Proof:__ $M \setminus (N \setminus M) = M \setminus (N \wedge (N \wedge M)^{\perp}) = M \wedge (M \wedge N \wedge (N \wedge M)^{\perp})^{\perp}$
$= M \wedge \overline{((M \wedge N) \wedge (M \wedge N)^{\perp})^{\perp}} = M \wedge 0^{\perp} = M \wedge H = M.$

__Theorem 19:__ $(L(H), \setminus)$ satisfies OS2.

__Proof:__ $M \setminus (M \setminus N) = (M \setminus N)^{\perp}_{M} = ((M \wedge N)^{\perp}_{M})^{\perp}_{M} = M \wedge N = N \setminus (N \setminus M)$ by symmetry.

__Theorem 20:__ $(L(H), \setminus)$ satisfies the orthomodular law.

__Proof:__ Let $M \leq N$. Also $M \leq M \vee N^{\perp}$. Thus $M \leq N \wedge (M \vee N^{\perp})$. To prove containment the other way note that $N^{\perp} \leq M^{\perp}$. Thus $N^{\perp} \perp M$ and $(M \vee N^{\perp}) = \{z + y | z \in M,\ y \in N^{\perp}\}$. Let $x \in N \wedge (M \vee N^{\perp})$, then $x = z + y \in N$ and hence $x - z = y \in N$. But $y \in N^{\perp}$ so that $y = 0$ and $x = z$. Therefore $x \in M$.

__Theorem 21:__ $(L(H), \setminus)$ satisfies OS3.

__Proof:__ Let $D = C \setminus (M \setminus N)$, hence $D \leq C$ and $D = D \wedge C$. $D \setminus M = (D \wedge M)^{\perp}_{D}$
$= (D \wedge C \wedge M)^{\perp}_{D} = (D \wedge (C \wedge M))^{\perp}_{C} \wedge D$ by compatibility. By DeMorgan's Law this is equal to $(D^{\perp}_{C} \vee (C \wedge M)^{\perp}_{C}) \wedge D$.

Since $M \setminus N \leq M$ it follows from the antitone property of $^{\perp}$ that $(C \wedge M)^{\perp}_{C}$
$= C \setminus M \leq C \setminus (M \setminus N) = D$. Thus the orthomodular law can be applied. Hence $(D^{\perp}_{C} \vee (C \wedge M)^{\perp}_{C}) \wedge D = (C \wedge M)^{\perp}_{C}$ and therefore $D \setminus M = (C \setminus (M \setminus N)) \setminus M = C \setminus M$, verifying OS3.

It follows that $(L(H), \setminus)$ is an orthosubtraction algebra.

Attention is now turned to the set of projection operators on H, P(H).

__Theorem 22:__ P(H) is a partially ordered set under $P \leq Q$ if and only if $P = PQ = QP$.

__Proof:__ (a) $P = P^2$, thus $P \leq P$ (reflexive),

(b) Suppose $P \leq Q$ and $Q \leq P$, then $P = PQ = PQP = P^2Q = PQ = QP = Q$ (Antisymmetric).

(c) Suppose $P \leq Q$ and $Q \leq R$, then $P = PQ = PQR = PR$ and thus $P \leq R$ (transitive), Lemma 2: If $P^{\perp}$ is defined as the projection onto the null space of P then $P^{\perp}P = PP^{\perp} = 0$.

Proof: Let $x \in H$, then $x = y + z$ where $y \in$ Range (P) and $z \in$ Null (P), thus $P^{\perp} Px = P^{\perp}y = 0$, as $y \perp$ Null (P). Also, $PP^{\perp}x = Pz = 0$ as $z \in$ Null (P).

Lemma 3: $P^{\perp} = I-P$.

Proof: Let $x \in H$, then $x = y + z$ where $y \in$ Range (P) and $z \in$ Null (P). $P^{\perp}x = z = (y+z)-y = x - y = Ix - Px = (I-P)x$, therefore $P^{\perp} = I-P$.

Theorem 23: P(H) is orthocomplemented.

Proof: (a) Suppose $Q \leq P$, $Q \leq P^{\perp}$, thus $Q = PQ = PP^{\perp}Q = 0Q = 0$ therefore glb $\{P, P^{\perp}\} = 0$.

(b) Suppose $P \leq R$, $P^{\perp} \leq R$, then $P^{\perp} = P^{\perp}R = (I-P)R = R - PR = R-P$. Since $P^{\perp} = I - P$ it follows that $I-P = R-P$ and hence $I = R$.

(c) $P^{\perp\perp} = I - P^{\perp} = I - (I-P) = P$.

Theorem 24: (Foulis) glb $\{P,Q\} = P \wedge Q = (P^{\perp}Q)^{\perp}Q$.

Corollary 1: lub $\{P,Q\} = P \vee Q = (P^{\perp} \vee Q^{\perp})^{\perp}$

Corollary 2: P(H) is a lattice.

Theorem 25: P(H) is an orthosubtraction algebra where $P\setminus Q$ is the projection onto Range $(P)\setminus$Range (Q).

Proof: OS1 through OS3 are verified by noting that L(H) is an orthosubtraction algebra.

Corollary: P(H) is an orthomodular lattice.

Note that the definition of the orthosubtraction is generally unwieldy to express in terms of composition of operators. If P and Q commute, though, it is easier to express $P \wedge Q$ and $P\setminus Q$.

Theorem 26: $P \wedge Q = PQ$ if and only if $PQ = QP$.

Proof: Suppose $P \wedge Q = PQ$. Then also $Q \wedge P = QP$. Since $\wedge$ is commutative $P \wedge Q = Q \wedge P$ and hence $PQ = QP$.

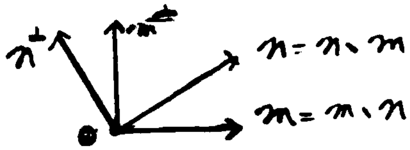
Suppose $PQ = QP$, then $(P^{\perp}Q)^{\perp}Q = ((I-P)Q)^{\perp}Q = (Q-PQ)^{\perp}Q = (I - Q + PQ)Q = Q - Q^2 + PQ^2 = PQ = P \wedge Q$.

Theorem 27: $P\setminus Q = P - PQ$ if and only if $PQ = QP$.

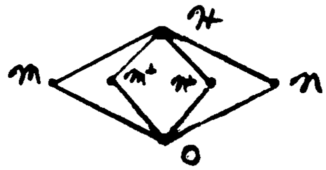
Proof: $P\setminus Q = P \wedge (P \wedge Q)^{\perp} = P \wedge (PQ)^{\perp} = P \wedge (I - PQ) = P(I - PQ)$ as $P(I - PQ) = P - PQ = P - P^2Q = P - PQP = (I - PQ)P$. Thus $P\setminus Q = P - P^2Q = P - PQ$.

To close this section the following are examples of orthosubtraction algebra as applied to Hilbert space:

Example 1: Let H be Euclidean 2-space, $\mathbb{R}^2$. The picture below shows the locations of various subspaces associated with two subspaces M and N:



The lattice diagram is below:



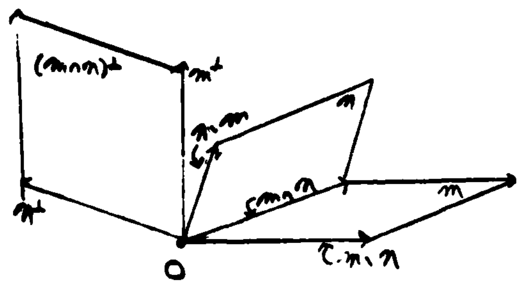
This is OM6, the smallest non-Boolean orthomodular lattice. To show that it is non-distributive look at $(M \vee M^{\perp}) \wedge N$ and $(M \wedge N) \vee (M^{\perp} \wedge N)$.

$$(M \vee M^{\perp}) \wedge N = H \wedge N = N$$

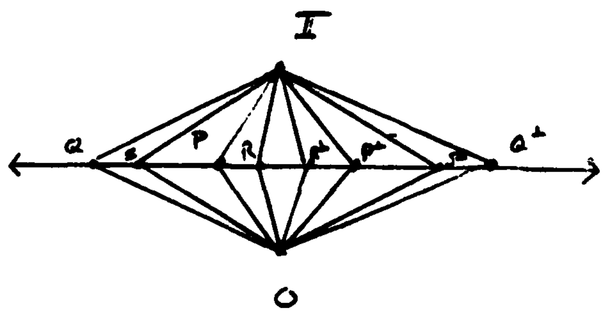
$$(M \wedge N) \vee (M^{\perp} \wedge N) = 0 \vee 0 = 0$$

Thus $(M \vee M^{\perp}) \wedge N \neq (M \wedge N) \vee (M^{\perp} \wedge N)$ and hence the lattice is non-distributive.

Example 2: Let H be Euclidean 3-space, $\mathbb{R}^3$. The picture below shows the locations of various subspaces associated with two subspaces M and N.



Example 3: Let $H = \mathbb{C}^2$, complex 2-space. The lattice diagram of P(H) is



CHAPTER III: MANUALS AND DASBAMS

PART ONE: Empirical Logic and Manuals

What is a manual? Most people usually think of a manual as a group of instructions which tell one how to operate a piece of equipment. An example of this is the owner's manual for an automobile or calculator. Another type of manual might specify procedures for using a system. One of the most important examples of this type of manual is the Reactor Plant Manual, which is virtually a Bible for operations with naval reactors on board nuclear submarines, surface ships, or prototype units. A third type of manual, also encountered in the Navy, is one which describes the duties a watchstander should perform and procedures for performing them. For example, a seaman standing lookout on a destroyer can refer to NAVEDTRA "Seaman" for an explanation of how to properly search for and report contacts. Finally, anyone who has taken any lab science course is familiar with laboratory manuals, which provide experiments to be performed and procedures for performing them.

Looking back at those four different types of manuals it is apparent that there is a common quality that they all share. This is the fact that they all tend to contain collections of operations, experiments, or tests to be performed by the user. These tests have outcomes to be noted by the experimenter. Furthermore, the performing of one test often interferes with the performing of another. For example, the lookout can search the horizon for surface contacts or he can search the sky for air contacts but he cannot do both simultaneously.

In many respects this is similar to problems encountered in the study of quantum mechanics. In the realm of the subatomic Heisenberg's Uncertainty Principle interferes and makes it impossible to perform a grand canonical operation in every case. The best that can be accomplished is to have a collection of operations which may not be simultaneously performable and which may interfere with each other.

It is the idea of a family of operations that have motivated Foulis and Randall of the University of Massachusetts to pursue an operational approach to empirical logic. Stated simply, empirical logic is an attempt to model mathematically the picture of "reality" presented by our senses, i.e., the outcomes of the experiments we perform. It avoids questions of existence such as "Does this pen have an existence independent of my sensation of it?" Instead it is concerned with data such as "When observed by the human eye this pen appears green."

The origins of empirical logic date back to 1846 and the British mathematician George Boole. In searching for a mathematical foundation for logic he invented the structure known as the Boolean algebra. A Boolean algebra is an algebra with the operations of disjunction, conjunction, and negation which correspond to the logical connective "or," "and," and "not." It has the property that it satisfies the distributive laws, which state that (A or B) and C is equivalent to (A and C) or (B and C) and also (A and B) or C is equivalent to (A or C) and (B or C). This same structure also serves as a model for set theory which has the operations of union, intersection, and complement. In 1936 Komolgorov formalized probability theory using Boolean algebra as a basis. The way this was accomplished was by defining a measure on the algebra which was disjointly additive, had a value of 1 on the greatest element (corresponding to A or not A) and 0 on the least element (corresponding to A and not A). This today is used as the basis for classical probability and statistics as taught in high school or college (such as SM239 at the Naval Academy). As explained in that course there is an experiment such as flipping a coin or rolling a die which generates a sample space of outcomes. Events can be considered as sets of outcomes. A probability function is defined on the sample space which assigns a weight from 0 to 1 on each outcome and which sums to 1 on all the outcomes. This weighting can be extended to the events by taking the weight of an event to be the sum of all of the weights of the outcomes contained in it.

Example 1: Let the experiment consist of the rolling of a die. The sample space is $\{1,2,3,4,5,6\}$ and a probability function could be one which assigns weight 1/6 to each outcome. Take $\{2,4,6\}$ to be an event, then the weight of $\{2,4,6\}$ is 1/2.

The methods of classical statistics work well when there is just one experiment. What happens, however, when there is more than one experiment? Classical statistics does not work so well in this case and it is beneficial to use "non-classical" statistics. It is because of this situation that Foulis and Randall formulated the concept of a manual. This concept and other related concepts will be rigorously mathematically defined later, but first an intuitive explanation. As the examples at the beginning illustrated, a manual is a collection of experiments or operations to obtain outcomes. In general there is no biggest operation containing the others. Each operation, however, is an experiment in the classical probability sense with a sample space of outcomes and a related set of events. Outcomes from different experiments which represent the same property are identified and said to be the same outcome. In this case the operations can be said to overlap.

With these ideas in mind the mathematical definitions of Foulis and Randall are now presented.

Definition 1:[6] (a) A premanual $\mathcal{A}$ is a non-empty set of non-empty sets, E,F,G, etc., where $E = \{a,b,c,...\}$ is called an operation, test or experiment.

(b) $\mathcal{A}$ is irredundant if and only if $E \subseteq F$ implies that $E = F$.

(c) $a \in E$ is called an outcome or atom.

(d) $A \subseteq E$ is called an event.

(e) $X = \bigcup_{E \in \mathcal{A}} E$ is the set of all outcomes of $\mathcal{A}$

(f) The set of all events is denoted $\mathcal{E}(\mathcal{A})$

(g) If A and B are events then A oc B if and only if $A \cup B = E$ where $E \in \mathcal{A}$, and $A \cap B$ is empty. A and B are called operational complements.

(h) A op B if and only if there is a common operational complement C with A oc C and C oc B. A and B are said to be operationally perspective.

(i) $A \perp B$ if and only if $A \cap B = \emptyset$ and $A \cup B$ is an event. A and B are said to be orthogonal.

The motivation for these definitions follows from the intuitive ideas. Any good set of instructions should be irredundant for the sake of efficiency. The outcomes are the consequences of the tests performed and are what will be recorded in the operators log. Events are sets of outcomes as in classical probability. Operational complements are just that, complements within an operation. If A oc B and the operation is performed then either A or B but not both will occur. When considered in the context of Hilbert spaces oc can also be thought of as representing orthogonal complementation. Events which are operationally perspective confirm one another, that is if A oc C and B oc C then when A occurs C does not and hence if the operation $B \cup C$ were performed then B would occur. The term arises from projective geometry and in the context of Hilbert space can be interpreted as "span the same subspace." Finally, if $A \perp B$ then if A occurs B does not and the term again is motivated by Hilbert space.

Condition M: A op B and B oc C implies A oc C.

Definition 2: A manual is an irredundant premanual which satisfies Condition M.

The requirement for Condition M is desirable by thinking of a manual as a set of instructions. If A confirms B and B rejects C it makes sense that A should reject C. Condition M requires that there be a test which directly establishes the fact that A rejects C.

This idea of a manual does indeed fit in with the idea of a manual as explained previously.

PART TWO: Application of Subtraction Algebra to Maunuals

In Chapter I subtraction algebras were axiomatically developed and shown to be identical to semi-Boolean algebras. Semi-Boolean algebras are useful in that they have most of the properties of Boolean algebras but are not as restrictive as there is not, in general, a greatest element. The application of this structure in either subtraction or semi-Boolean form shall be illustrated in this section.

Definition 3: (a) A semi-Boolean (subtraction) algebra $(S,\setminus)$ is atomic if and only if there exists a subset $A \subseteq S$ of atoms satisfying two conditions:

A1 For all non-zero x in S there exists $a \in A$ with $a \leq x$.

A2 For all $x \in S$, $a \in A$, $x \leq a$ implies $x = 0$ or $x = a$.

(b) The algebra is dominated if and only if there exists a subset $M \subseteq S$ of maximal elements satisfying these two conditions:

D1 For $x \in S$ there exists $e \in M$ with $x \leq e$.

D2 For all $x \in S$, $e \in M$ if $e \leq x$ then $e = x$.

These are terms from lattice theory. In general, an atom is a minimal non-zero element. If every element has at least one atom contained in it then the algebra is atomic. In a Boolean algebra these properties imply that every element is the join of all atoms beneath it. When this is the case then the algebra is atomistic. In a semi-Boolean algebra it is again true that atomic implies atomistic. It is not generally true for lattices. For example, in the lattice of natural numbers under the usual order 0 is at least element and 1 the only atom. A1 and A2 are true but no element except 1 is a join of atoms.

Given a dominated atomic semi-Boolean algebra other relations can be defined.

Definition 4: (a) x oc y if and only if $x \wedge y = 0$ and $x \vee y = e$ where $e \in M$.

(b) x op y if and only if there exists z such that x oc z and z oc y.

(c) $x \perp y$ if and only if $x \wedge y = 0$ and $x \vee y$ exists.

Hence x oc y means that x and y are relative complements within some Boolean principal ideal generated by a maximal element or operation. We call x and y operational complements. Since x may be dominated by more than one maximal element it is possible that x may have distinct operational complements. The relation op, operational perspectivity, symbolizes this fact. The motivation of the term orthogonal will be evident later.

Lemma 1: (a) x oc y if and only if there exists $e \in M$ with $x \leq e$, $y \leq e$ and $x = e \setminus y$.

(b) x op y if and only if there exist e and f in M with $x \leq e$, $y \leq f$, and $e \setminus x = f \setminus y$

This lemma allows expression of the relations in terms of subtraction.

Theorem 1: $\mathcal{Q}$ is an irredundant premanual if and only if the event structure $\mathcal{E}(\mathcal{Q})$ is a dominated atomic semi-Boolean algebra.

Proof: (a) Let $\mathcal{Q}$ be an irredundant premanual and define subtraction on $\mathcal{E}(\mathcal{Q})$:

$A \setminus B = \{a \in A | a \notin B\}$

The axioms need to be verified.

(S1) $A \setminus (B \setminus A) = \{a \in A | a \notin (B \setminus A)\} = \{a \in A | a \notin B \text{ or } a \in A\} = \{a \in A\} = A.$

(S2) $A \setminus (A \setminus B) = \{a \in A | a \notin (A \setminus B)\} = \{a \in A | a \notin A \text{ or } a \in B\} = \{a \in A | a \in B\} = A \cap B = B \setminus (B \setminus A)$

(S3) $(A \setminus B) \setminus C = \{a \in A \setminus B | a \notin C\} = \{a \in A | a \notin B \text{ and } a \notin C\} = \{a \in A | a \notin C \text{ and } a \notin B\} = (A \setminus C) \setminus B.$

(A1) Let $A \in \mathcal{E}(a) (A \neq \emptyset)$. Since A is an event there is some outcome $a \in A$, thus $\{a\} \subseteq A$.

(A2) Suppose $A \subseteq \{a\}$, then either $\{a\} = A$ or $A = \emptyset$.

(D1) Let A be an event, then there exists $E \in \mathcal{Q}$ with $A \subseteq E$.

(D2) Suppose $E \subseteq A$. By irredundance $E = A$. Therefore $\mathcal{E}(\mathcal{Q})$ is a dominated atomic semi-Boolean algebra.

(b) Let $(S, \setminus, A, M)$ be a dominated atomic semi-Boolean algebra. For each element x form A_x, the set of atoms beneath $x : A_x = \{a \in A | a \leq x\}$. For all $e \in M, A_e$ is a collection of atoms. Take $\mathcal{Q} = \{A_e\}$ indexed over M. By A1 A_e is non-empty and by D2 if $A_e \subseteq A_f$ then $A_e = A_f$. Therefore $\mathcal{Q}$ is an irredundant premanual.

Theorem 2: If $\mathcal{Q}$ is an irredundant premanual and $\mathcal{E}(\mathcal{Q})$ is the associated event structure then the relation of oc on $\mathcal{E}(\mathcal{Q})$ is the same whether considered as an event structure or as a DASBA.

Proof: Take $\mathcal{E}(\mathcal{Q})$ as an event structure and suppose A oc B. $A \cap B = \emptyset$ and $A \vee B = E$ implies that $A = E \setminus B$ and thus A oc B in the sense of a DASBA. The converse is also true.

From this result the relation of op can also be seen to be the same in the irredundant premanual sense as in a DASBA because it is derived from the oc relation. The following result also holds.

Theorem 3: $\mathcal{Q}$ is a manual if and only if $\mathcal{E}(\mathcal{Q})$ is a dominated atomic semi-Boolean algebra satisfying Condition M (a DASBAM).

Proof: Since the relations of oc and op in each structure correspond Condition M must also correspond since it is impressed only in terms of oc and op.

This last theorem states the major difference between the viewpoint of Abbott and the Naval Academy group and that of Foulis and Randall and the Amherst group. In Amherst, the manual is studied mainly at the top (the operations) and the bottom (the outcomes). In Annapolis, the entire semi-Boolean event structure is considered and while the operations and atoms are important it is not to the exclusion of the rest of the structure. By thinking in terms of a DASBAM the idea is one of an algebraic structure whereas the idea of a manual suggests a set theoretic approach.

PART THREE: Properties and Types of DASBAMs.

The simplest type of DASBAM is one like Example 1 in Part One of this chapter, the set of events of a classical sample space. In this example, the atoms are the singleton events corresponding to the outcomes and the set of maximal elements has only one member, corresponding to the experiment. Since the op relation is trivial Condition M holds automatically. This type of DASBAM where there is only one operation is called a classical DASBAM. It is a Boolean algebra.

Almost as simple as a classical DASBAM is a semi-classical DASBAM, defined by $e \setminus f = e$ whenever $e \in M$, $f \in M$, and $e \neq f$. The tests do not overlap each other, having only 0 in common. If each test is atomic and dominated, then so is the whole algebra. To see that Condition M holds note that the only non-trivial op pairs occur among the tests and the common complement is 0. Thus if e op f and f oc x then $x = 0$ and hence e oc x.

Other types of DASBAMs are not as easy to explain. There are, however, some theorems which determine whether or not a given semi-Boolean algebra is a DASBAM.

Theorem 4: If a DASBA has only two tests it satisfies Condition M.

Proof: Let S be a DASBA with $M = \{e,f\}$. Suppose x op y. There are two possibilities, either $x = y$ or else $x \neq y$. If $x = y$ then y oc z implies x oc z trivially. If $x \neq y$ then there is an element z with x oc z and z oc y. Without loss of generality assume $z = e \setminus x$ and $z = f \setminus y$. $z \leq e$ and $z \leq f$, hence $z \leq e \wedge f$. Since $f \setminus e = f \setminus (e \wedge f)$ it follows by the antitone law that $f \setminus e \leq f \setminus z = y$. Thus $y \geq e \setminus f$ and hence its only complement is z. Again y oc z implies x oc z.

For the next types of DASBAM it is necessary to introduce the concept of ghosting.[7]

Definition 5: Let P be a partially ordered set, S, a subset of P, and Q a partially ordered set with lower bound o. Define the ghosting of S within P by Q as follows: Form $S \times Q = \{(s,q) | s \in S \text{ and } q \in Q\}$ and let $(s,q) \leq (s',q')$ if and only if $s \leq s$ and $q \leq q'$. Identify $s \in S$ with (s, o) and $p \in P \setminus S$ with (p,o). The new structure is a partially ordered set.

What is being done is the adjoining of a copy of Q with 0 at s for every point $s \in S$. This is similar to an out-of-tune TV picture of P, which is known as a ghost. We will now specialize this idea for application to semi-Boolean algebras.

Definition 6: Let S be a DASBA and consider the series of ghostings of I(e) within S by B_2 where e runs over the dominating set. The resulting structure is called the dacification of S and is denoted S^+.[8] If I(e) is one such maximal principal ideal then let e^+ be the atom of B_2 by which I(e) is ghosted. The set $I(e) \times e^+ = \{(x,e^+) | x \leq e\}$ is defined to be the ghost of I(e).

At this point it is useful to illustrate these terms via an example.

<u>Example 2</u>: Let S be S({x,y}), the free subtraction algebra on two generators. S is a DASBAM. Below are shown S and S^+.

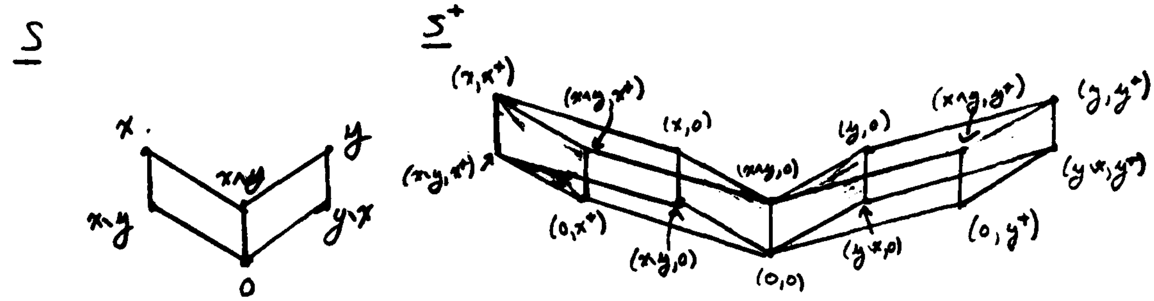


Figure 1

In the partial order diagram the ghost of I(y) is the set $\{(o,y^+), (x \wedge y, y^+), (y \smallsetminus x, y^+), (y, y^+)\}$.

In any dacification the ghosts of different tests do not overlap since $e^+ = f^+$ implies $e = f$. The atoms of the dacification are of the form (a,o) or (o, e^+) where a is an atom of S. The tests are of the form (e, e^+) where e is a test of S.

<u>Lemma 2</u>: x oc y in S^+ if and only if exactly one of the two is in S and the other in a ghost.

<u>Proof</u>: $(e, e^+) \smallsetminus x = (e \smallsetminus x_1, o)$ or $(e \smallsetminus x_1, e^+)$. If the first is true then x is in the ghost of I(e) and y is in S. If the second is true then x is in S and y is in the ghost of I(e).

<u>Lemma 3</u>: In S^+ if x oc y, y oc z, and z oc w then either x = z or y = w.

<u>Proof</u>: Suppose x is in a ghost, then $x = (x_1, e^+)$ and $y = (e \smallsetminus x_1, o)$. Since y oc z then z is in a ghost and $z = (f \smallsetminus (e \smallsetminus x_1), f^+)$ where $f \in M$. Hence $w = (f \smallsetminus (f \smallsetminus (e \smallsetminus x_1)), \phi) = (e \smallsetminus x_1, o) = y$.

Suppose x is in S, then $y = (e \smallsetminus x, e^+)$ and $z = (e, e^+) \smallsetminus y = (e \smallsetminus (e \smallsetminus x), e^+ \smallsetminus e^+) = (x, \phi) = x$.

<u>Theorem 5</u>: If S is a DASBA then S^+ is a DASBAM.

<u>Proof</u>: It merely needs to be shown that S^+ satisfies Condition M. Let x op z and z oc w. By definition of op there exists $y \in S^+$ with x oc y and y oc z. There are two cases: (a) x = z implies x oc w, thus Condition M holds; (b) y = w implies x oc w, thus Condition M holds.

The process of dacification produces a DASBAM from a DASBA. The next process produces a new DASBAM from two DASBAMs.

<u>Theorem 6</u>: If S and T are DASBAMs then S X T is a DASBAM.[9]

<u>Proof</u>: From the remarks following Definition 9 of Chapter I, S X T is a semi-Boolean algebra. A1, A2, D1, D2, and Condition M must now be verified.

For the atoms in the ghosting look at the set $\{(a,o) | a$ an atom of $S\}$ $\{(o,b) | b$ an atom of $T\}$.

<u>A1</u>: For all $x = (s,t) \neq (o,o)$ either there exists $a \in A_s$ with $a \leq s$ or if $s = o$ then there exists $b \in A_T$ with $b \leq t$. Thus $(a,o) \leq (s,t)$ or $(o,b) \leq (s,t)$.

<u>A2</u>: Suppose $x = (s,t) \leq (a,o)$ then $s \leq a$ and $t = o$. Hence $s = a$ or $s = o$ and $x = (a,o)$ or (o,o). Likewise $x = (s,t) \leq (o,b)$ implies $x = (o,b)$ or (o,o).

For the set of maximal elements consider the set $\{(e,f) | e \in M_s$ and $f \in M_T\}$.

<u>D1</u>: Let $x = (s,t)$, then there exists $e \in M_s$ with $s \leq e$ and $f \in M_T$ with $t \leq f$, thus $(s,t) \leq (e,f)$.

<u>D2</u>: Suppose $(e,f) \leq (s,t)$, then $e \leq s$ and $f \leq t$. Hence $e = s$ and $f = t$, thus $(e,f) = (s,t)$.

Finally, since subtraction is defined coordinatewise so are the relations of oc and op, that is

$$(s,t) \text{ oc } (s',t') \text{ if and only if } (s,t) = (e,f)\setminus(s',t')$$

if and only if $s = e\setminus s'$ and $t = f\setminus t'$ if and only if s oc s' and t oc t'.

Likewise (s,t) op (s',t') if and only if s op s' and t op t'.

<u>M</u>: Hence (s,t) op (s',t') and (s',t') oc (s",t") implies s op s', s' oc s", t op t', t' oc t". Hence s oc s" and t oc t". Therefore (s,t) oc (s",t").

<u>Theorem 7</u>: If S is a free subtraction algebra with a finite set of generators $\{e,f,g...\}$ then S is a DASBAM[10]

<u>Proof</u>: Since S is finite it is atomic. The set of generators is the dominating set. Finally, note that S is a dacification since given a test e form the element $(e\setminus f)\setminus g..=x$. This element is contained in e alone. I(e) is isomorphic to $I(e\setminus x) \times I(x)$.

This section has given some methods for recognizing DASBAMs and for constructing new DASBAMs. In the next section some examples will be presented.

PART FOUR: Examples of DASBAMs and Manuals

Example 3: Let B be a Boolean algebra which has a greatest element 1 and is atomic. The oc relation is merely complementation and the op relation is just equality, hence Condition M holds. This is a classical DASBAM.

Example 4: Let $\mathfrak{A} = \{\{a,b\}, \{c,d\}, \{e,f\}\}$. Since there is no overlap between operations $\mathfrak{A}$ is a semiclassical manual and $E(\mathfrak{A})$ is a semi-classical DASBAM. The partial order diagram is below:

$E(\mathfrak{A})$

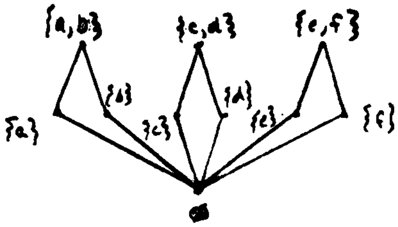


Figure 2

Example 5: Let S be the free subtraction algebra on two generators, x and y. Since there are only two tests S is a DASBAM. The partial order diagram is Figure 1 of Chapter I.

Example 6: Let S be as shown below. S is a DASBA. However, by looking at x, z, and w as in the diagram it is easy to note that x op z and z oc w but there is no operation so that x oc w.

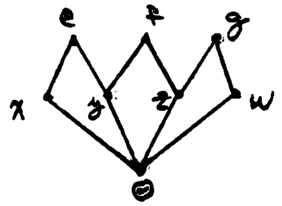


Figure 3

Example 7: Let S be as in Example 5 and form the direct product S X B_2. The initial and final partial order diagrams are as in Figure 1.

Example 8: Let S be as in Example 5 and form S X T where T is as shown below. This direct product is a DASBAM.

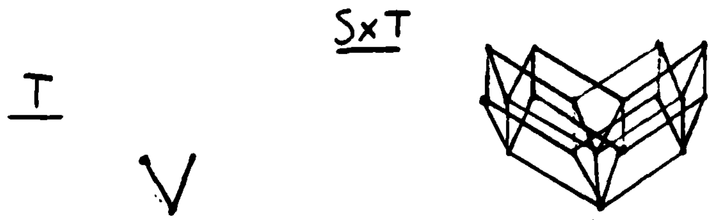


Figure 4

Example 9: Let S be as in Example 4 and form the dacification of S, S+. The partial order diagram is given below.

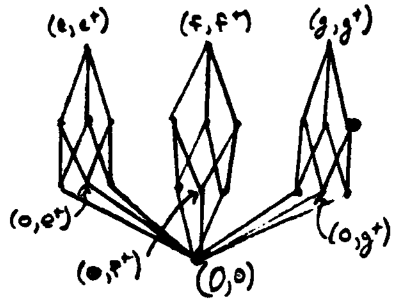


Figure 5

Although S was not a DASBAM S+ is.

Example 10: (The Lookout's Manual)[11]

Let $\mathcal{A} = \{\{a,b,c\}, \{c,d,e\}, \{e,f,a\}\}$. The partial order diagram of $E(\mathcal{A})$ is below.

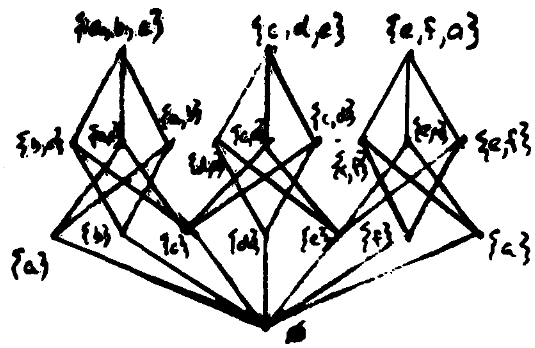


Figure 6

By inspection this is a dacification of $\mathcal{A}_0 = \{\{a,c\}, \{c,e\}, \{e,a\}\}$ and thus $E(0)$ is a DASBAM.

This example is known as the lookout's manual because it arises from a situation in navigation. Look at Figure 7:

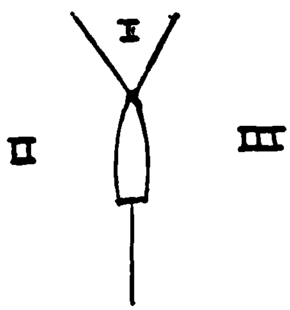


Figure 7

A vessel is steaming in the open ocean with three lookouts posted; a port lookout, a starboard lookout, and an after lookout. Each lookout can make reports as listed below:

Post Lookout	a - contact in sector I b - no contact visible c - contact in sector II
Starboard Lookout	a - contact in sector I f - no contact visible e - contact in sector III
After Lookout	c - contact in sector II d - no contact visible e - contact in sector III

Each lookout represents a test abailable to the Officer of the Deck with outcomes as above. The collection of tests forms a manual and the event structure is a DASBAM.

<u>Example 12</u>: Let H be a Hilbert space. Take $\mathcal{Q}$ to be the set of all orthonormal bases of H. The atoms are the unit vectors of H and the events are sets of orthogonal unit vectors.

Suppose that $A \underline{\perp} B$ as events, then $A \cup B$ is a subset of an orthonormal basis of H. Hence $a \perp b$ for all $a \in A$ and $b \in B$ and $A \perp B$ as sets of vectors in a Hilbert space. Therefore $\underline{\perp}$ in the DASBAM and $\perp$ in the Hilbert space correspond and the term "orthogonal" is justified.

Let A oc B, then $A \underline{\perp} B$ and $A \perp B$. Also $A \cup B$ is an orthonormal basis. Hence $(\text{Span }(A))^{\perp} = \text{Span }(B)$ and A and B can be thought of as orthogonal complements. Furthermore, since $(\text{Span }(A))^{\perp}$ is unique, Span (C) = Span (B) whenever C oc A. Thus C op B is the same as the relation "Spans the same subspace as ."

The verification of Condition M in this case is now easy. A op B and B oc C imply that Span (A) = Span (B) and Span (B) = $(\text{Span }(C))^{\perp}$. Thus Span (A) = $(\text{Span }(C))^{\perp}$ and therefore A oc C.

CHAPTER IV: THE OP LOGIC

PART ONE: Properties of the OP Logic

In the last chapter the requirements for a semi-Boolean algebra to be atomic and dominated made sense when considering DASBAMs and manuals for use in empirical logic. The requirement for Condition M might have less obvious a motivation.

<u>Theorem 1</u>: If Condition M holds then the op relation is transitive.

<u>Proof</u>: Let x op y and y op z. y op z implies there exists w such that y oc w and z oc w. By Condition M, x oc w, thus w is a common complement of x and z. Therefore x op z.

<u>Theorem 2</u>: If Condition M holds then op is an equivalence relation.

<u>Proof</u>: (1) x op x is obvious (Reflexive)

(2) x op y if and only if y op x is true by symmetry of the definition of op. (Symmetric)

(3) x op y and y op z if and only if x op z was proved in Theorem 1. (Transitive)

Since a DASBAM is an algebraic structure and op is an equivalence relation it makes sense to investigate the quotient structure of the algebra modulo this equivalence relation. This will be called the op logic of the DASBAM.

To get a feel for what is happening in terms of the structure of the DASBAM the following results will be established.

<u>Lemma 1</u>: If x op y via common complement z where $x \leq e$, $y \leq f$, and $e \setminus x = z = f \setminus y$ the following are true.

(a) $z \leq e \wedge f$

(b) $e \setminus f \leq x$ and $f \setminus e \leq y$

(c) $x \wedge y \leq e \wedge f$

(d) $x \wedge y = (e \wedge f) \setminus z$

(e) $x \setminus y = e \setminus f$ and $y \setminus x = f \setminus e$

<u>Proof</u>: (a) Since $z = e \setminus x$ and $e \setminus x \leq e$, it follows that $z \leq e$. Also $z = f \setminus y$ and $f \setminus y \leq f$ imply that $z \leq f$. Hence $z \leq e \wedge f$.

(b) $z \leq e \wedge f$. Thus by the antitone law $e \setminus (e \wedge f) \leq e \setminus z$. Therefore $e \setminus f \leq x$. Without loss of generality $f \setminus e \leq y$.

(c) $x \leq e$ and $y \leq f$. Thus $x \wedge y \leq e \wedge f$.

(d) First $(x \wedge y) \wedge (z) = (x \wedge (e \setminus x)) \wedge y = o \wedge y = o$. Second $(x \wedge y) \vee (z) = ((e \setminus x) \vee x) \wedge ((f \setminus y) \vee y) = e \wedge f$. Therefore z and $x \wedge y$ are relative complements in $I(e \wedge f)$ and hence $x \wedge y = (e \wedge f) \setminus z$.

(e) $e = (e \setminus f) \vee (e \wedge f) = (x \wedge y) \vee z \vee (e \setminus f)$ where the terms are pairwise disjoint. Hence $e \setminus z = e \setminus (e \setminus x) = x = (e \setminus f) \vee (x \wedge y)$. Thus $x \setminus y = x \setminus (x \wedge y) = e \setminus f$ and without loss of generality $y \setminus x = f \setminus e$.

These facts are illustrated in Figure 1.

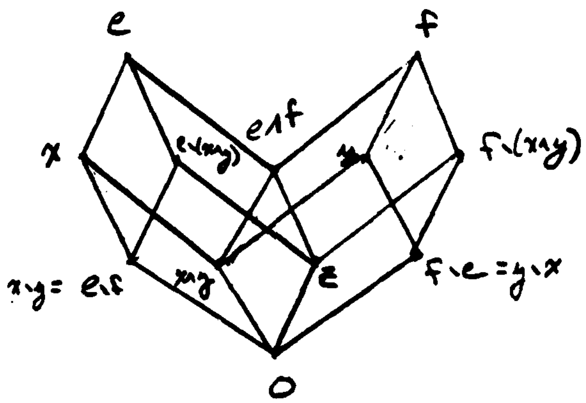


Figure 1

When x op y their common complement lies in $I(e \wedge f)$. For this reason $I(e \wedge f)$ is called the axis of perspectivity.

The op-logic shall now be defined as a quotient structure of the DASBAM modulo the op relation.

<u>Definition 1</u>: (a) $\overline{x} = \{y \mid y \text{ op } x\}$

(b) $\overline{S} = \{\overline{x} \mid x \in S\}$ is the op-logic of the DASBAM S.

It is desirable to determine what sort of structure the logic has and what properties it inherits from the DASBAM. To begin a partial order relation is defined.

<u>Definition 2</u>: $\overline{x} \leq \overline{y}$ if and only if for all $x_1 \in \overline{x}$ there exists $y_1 \in \overline{y}$ with $x_1 \leq y_1$.

<u>Lemma 2</u>: If x_1 op x_2 and there exists $e \in M$ with $x_1 \leq e$ and $x_2 \leq e$ then $x_1 = x_2$.

<u>Proof</u>: x_1 oc $e \setminus x_1$, thus x_2 oc $e \setminus x_1$. Hence $f \setminus x_2 = e \setminus x_1$ and $x_2 \vee (e \setminus x_1) = f$ where $f \in M$. Since $x_2 \leq e$ and $e \setminus x_1 \leq e$ it follows that $f \leq e$. Thus $f = e$. Since relative complements are unique $x_1 = x_2$.

With this lemma it can now be established that $\leq$ is a partial order relation on $\overline{S}$.

<u>Theorem 3</u>: $(\overline{S}, \leq)$ is a partially ordered set.

<u>Proof</u>: (1) $\overline{x} \leq \overline{x}$ is clear from the definition (Reflexive)

(2) Suppose $\overline{x} \leq \overline{y}$ and $\overline{y} \leq \overline{x}$. Then for all $x_1 \in \overline{x}$ there exists $y_1 \in \overline{y}$ with $x_1 \leq y_1$. But for $y_1 \in \overline{y}$ there exists $x_2 \in \overline{x}$ with $y_1 \leq x_2$. Thus $x_1 \leq x_2$. But x_1 op x_2 and thus $x_1 = x_2 = y_1$. Therefore $\overline{x} = \overline{y}$. (Anti-symmetric)

(3) Suppose $\overline{x} \leq \overline{y}$ and $\overline{y} \leq \overline{z}$. For all $x_1 \in \overline{x}$ there is $y_1 \in \overline{y}$ with $x_1 \leq y_1$. Also since $\overline{y} \leq \overline{z}$ there exists $z_1 \in \overline{z}$ with $y_1 \leq z_1$. Hence $x_1 \leq z_1$ and $\overline{x} \leq \overline{z}$. (Transitive)

This partial order is essentially the same partial order from the DASBAM lifted to the logic. The next theorem gives an alternate definition of this order.

<u>Theorem 4</u>: $\overline{x} \leq \overline{y}$ if and only if there exist $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ with $x_1 \leq y_1$.

<u>Proof</u>: Suppose $\overline{x} \leq \overline{y}$, then the result follows obviously from Definition 2.

Suppose there exist $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ with $x_1 \leq y_1$. Let x_2 op x_1, it must be shown that there exists $y_2 \in \overline{y}$ with $x_2 \leq y_2$. Since x_1 op x_2 there exist e and f with $e \smallsetminus x_1 = f \smallsetminus x_2$. By the anitone law $e \smallsetminus y_1 \leq e \smallsetminus x_1$, thus $e \smallsetminus y_1 \leq f \smallsetminus x_2$. Let $y_2 = f \smallsetminus (e \smallsetminus y_1)$. By the antitone law $f \smallsetminus (f \smallsetminus x_2) \leq f \smallsetminus (e \smallsetminus y_1)$. But $x_2 \leq f$ and this $x_2 = f \wedge x_2 = f \smallsetminus (f \smallsetminus x_2) \leq y_2$. Also $e \smallsetminus y_1 = f \smallsetminus y_2$, so that y_1 op y_2. Thus for all $x_2 \in \overline{x}$ there is $y_2 \in \overline{y}$ with $x_2 \leq y_2$ and hence $\overline{x} \leq \overline{y}$.

This second definition of the partial order is the easier of the two to use when checking whether or not $\overline{x} \leq \overline{y}$.

The next step is to define an orthocomplementation on $\overline{S}$.

<u>Definition 3</u>: $\overline{x}^{\perp} = \{y | y \text{ oc } x\}$

From Condition M it doesn't matter which $x \in \overline{x}$ is chosen for the definition.

<u>Theorem 5</u>: $\overline{x}^{\perp} \in \overline{S}$.

Proof: It must be shown that $\overline{x}^{\perp} = \overline{y}$ for some y. Let $y_1 \in \overline{x}^{\perp}$ and $y_2 \in \overline{x}^{\perp}$. Thus y_1 oc x and y_2 oc x. Therefore y_1 op y_2. Also, suppose y_1 op y_2 and $y_1 \in \overline{x}^{\perp}$. Then y_1 oc x and by Condition M, y_2 oc x. Hence $y_2 \in \overline{x}^{\perp}$ and $\overline{y}_1 = \overline{x}^{\perp}$.

Theorem 6: $(\overline{x}^{\perp})^{\perp} = \overline{x}$.

Proof: $(\overline{x}^{\perp})^{\perp} = \overline{y}^{\perp}$ where y oc x.

$\overline{y}^{\perp} = \{z | z \text{ oc } y\}$ and thus z oc y and y oc x. Hence z op x and $\overline{z} = (\overline{x}^{\perp})^{\perp} = \overline{x}$.

Theorem 7: $(\overline{S}, \leq)$ is bounded above and below.

Proof: (above) Let $M = \overline{1}$. For all $\overline{x} \in \overline{S}$, and for all $x_1 \in \overline{x}$ there is $e \in \overline{1}$ with $x_1 \leq e$. Hence $\overline{x} \leq 1$.

(below) Since $0 \leq x$ for all x in S it follows that $\overline{0} \leq \overline{x}$ for all $\overline{x} \in \overline{S}$.

Theorem 8: $\overline{0}$ is the greatest lower bound of $\overline{x}$ and $\overline{x}^{\perp}$.

Proof: Suppose there exists $\overline{z}$ with $\overline{z} \leq \overline{x}$ and $\overline{z} \leq \overline{x}^{\perp}$, then for all $z_1 \in \overline{z}$ there exist $x_1 \in \overline{x}$, $y_1 \in \overline{x}^{\perp}$ with $z_1 \leq x_1$, $z_1 \leq y_1$, and x_1 oc y_1. Hence $z_1 \leq x_1 \wedge y_1$. But $y_1 \wedge x_1 = 0$. Thus $z_1 = 0$ and $\overline{z} = \overline{0}$

Theorem 9: 1 is the least upper bound of $\overline{x}$ and $\overline{x}^{\perp}$.

Proof: Suppose there exists $\overline{z}$ with $\overline{x} \leq \overline{z}$ and $\overline{x}^{\perp} \leq \overline{z}$. For all $x_1 \in \overline{x}$ there exists $z_1 \in \overline{z}$ with $x_1 \leq z_1 \leq e$ where $e \in M$. Hence $e \setminus z_1 \leq e \setminus x_1 \leq e$. $\overline{x}^{\perp} \leq \overline{z}$ implies that there exists $z_2 \in \overline{z}$ with $e \setminus x_1 \leq z_2 \leq e$ and thus $e \setminus z_1 \leq z_2$. Hence $e \setminus z_1 = (e \setminus z_1) \wedge z_2 = 0$ and $z_1 = e$, therefore $z = 1$.

From the last 5 theorems it follows that $(\overline{S}, \leq, ^{\perp}, 0, 1)$ is a bounded orthocomplemented partially ordered set. In general, however, least upper bounds or greatest lower bounds do not exist for a pair of elements. This fact will be illustrated in the examples in Part Two.

Lemma 3: $\overline{x} \leq \overline{y}$ implies $|\overline{x}| \leq |\overline{y}|$ where $|\overline{x}|$ denotes the number of elements of $\overline{x}$.

Proof: Let $\overline{x} \leq \overline{y}$. For each $x \in \overline{x}$ there is at least one $y \in \overline{y}$ such that $x \leq y$. Define $Y_x = \{y \in \overline{y} | x \leq y\}$. Y_x is non-empty. By the Axiom of Choice a single $y = y(x)$ can be chosen from Y_x for each x. The resulting map $x \to y(x)$ is a mapping from $\overline{x}$ to $\overline{y}$. It remains to be shown that this map is one-to-one. Let $x_1, x_2 \in \overline{x}$ and suppose $y(x_1) = y(x_2)$. Then $x_1 \leq y(x_1)$ and $x_2 \leq y(x_1)$ and thus $x_1 = x_2$ by Lemma 2. Since there is an injection from $\overline{x}$ to $\overline{y}$ it follows that $|\overline{x}| \leq |\overline{y}|$.

This result shows that the cosets get larger in magnitude as one gets higher in the logic. Also, by Lemma 2 the equivalence classes are antichains as no two elements of a class are even contained in the same maximal element.

Lemma 4: x_1 op x_2 and $x_1 \perp y$ implies $x_2 \perp y$.

Proof: Let $x_1 \perp y$, then $y \leq e \setminus x_1$ where $x_1 \vee y \leq e$. By Condition M, x_1 oc $e \setminus x_1$ implies x_2 oc $e \setminus x_1$. Thus there exists $f \in M$ with $(e \setminus x_1) \vee x_2 = f$ and $(e \setminus x_1) \wedge x_2 = 0$.

However, $y \leq e \setminus x_1$ implies $y \wedge x_2 \leq (e \setminus x_1) \wedge x_2 = 0$, thus $y \wedge x_2 = 0$. Also $y \vee x_2 \leq (e \setminus x_1) \vee x_2 = f$. Therefore $y \vee x_2$ exists and hence $y \perp x_2$.

Lemma 5: x_1 op x_2 and $x_1 \perp y$ implies $x_1 \vee y$ op $x_2 \vee y$.

Proof: Since op respects $\perp$ it follows that $x_2 \perp y$ and that $x_2 \vee y$ exists. Let $x_1 \vee y \leq e$ and $x_2 \vee y \leq f$. Then $y \leq e \setminus x_1$ and $y \leq f \setminus x_2$.

$$e = x_1 \dot{\vee} (e \setminus x_1) = x_1 \dot{\vee} y \dot{\vee} [(e \setminus x_1) \setminus y] \quad \text{and}$$

$$f = x_2 \dot{\vee} (f \; x_2) = x_2 \dot{\vee} y \quad \dot{\vee} [(f \setminus x_2) \setminus y]$$

where $\dot{\vee}$ denotes the fact that the elements are disjoint.

$$e \setminus (x_1 \dot{\vee} y) = (e \setminus x_1) \setminus y = (f \setminus x_2) \setminus y = f \setminus (x_2 \dot{\vee} y)$$

Hence $x_1 \vee y$ op $x_2 \vee y$.

Theorem 10: x_1 op x_2, y_1 op y_2, $x_1 \perp y$, implies that $x_1 \vee y_1$ op $x_2 \vee y_2$.

Proof: By Lemma 5 $x_1 \vee y_1$ op $x_2 \vee y_1$. Also $x_2 \vee y$ op $x_2 \vee y_2$ for the same reason. Since op is transitive $x_1 \vee y_1$ op $x_2 \vee y_2$.

By Theorem 10 and Lemma 4 $\perp$ has the substitution property with respect to op, thus the $\perp$ relation can be lifted to the op logic.

Definition 4: $\bar{x} \perp \bar{y}$ if and only if there exist $x_1 \in \bar{x}$ and $y_1 \in \bar{y}$ with $x_1 \perp y_1$.

Again, this is the same relation from the DASBAM lifted to the logic.

Lemma 6: If $\bar{x} \perp \bar{y}$ then $\bar{0}$ is the greatest lower bound for $\bar{x}$ and $\bar{y}$.

Proof: Suppose $\bar{z} \leq \bar{x}$ and $\bar{z} \leq \bar{y}$, then for all $z_1 \in \bar{z}$ there exist $x_1 \in \bar{x}$ and $y_1 \in \bar{y}$ with $z_1 \leq x_1$ and $z_1 \leq y_1$. Hence $z_1 \leq x_1 \wedge y_1 = 0$ and thus $\bar{z} = \bar{0}$.

Lemma 7: $\bar{x} \perp \bar{y}$ if and only if $\bar{x} \leq \bar{y}^{\perp}$.

Proof: Let $\bar{x} \perp \bar{y}$, then $x_1 \vee y_1$ exists for all $x_1 \in \bar{x}$ and $y_1 \in \bar{y}$. Thus $x_1 \leq e \setminus y_1$ for some test e and $\bar{x} \leq \overline{e \setminus y_1} = \bar{y}^{\perp}$.

Conversely, suppose $\overline{x} \le \overline{y}$. Then for all $x_1 \varepsilon \overline{x}$ there exists $y_1 \varepsilon \overline{y}$ and $e \setminus y_1 \varepsilon \overline{y}^{\perp}$ with $x_1 \le e \setminus y_1$. Thus x_1 and y_1 have an upper bound and $x_1 \wedge y_1 = 0$. Therefore $x_1 \perp y_1$, and $\overline{x}. \perp \overline{y}$.

<u>Lemma 8</u>: $x \vee z$ op $y \vee z$, $x \perp z$, and $y \perp z$ implies x op y.

<u>Proof</u>: $x \vee z$ op $y \vee z$ implies that there exist $e, f \varepsilon M$ with $e \setminus (x \vee z) = f \setminus (y \vee z) = w$. Hence $w \le e \wedge f$. Also $z \le x \vee z \le e$ and $z \le y \vee z \le f$, thus $z \le e \wedge f$. Therefore $w \vee z \le e \wedge f$. $e = w \dot{\vee} x \dot{\vee} z$ and $f = w \dot{\vee} y \dot{\vee} z$. Furthermore, $x \wedge (w \vee z) = y \wedge (w \vee z) = 0$. Therefore $e \setminus x = w \vee z = f \setminus y$ and hence x op y.

<u>Theorem 11</u>: x_1 op x_2, y_1 op y_2, $x_1 \le y_1$, $x_2 \le y_2$ implies $y_1 \setminus x_1$ op $y_2 \setminus x_2$.

<u>Proof</u>: y_1 op y_2 implies $e \setminus y_1 = f \setminus y_2 = z$. $e = y_1 \vee z$ and $f = y_2 \vee z$. $y_1 = x_1 \vee (y_1 \setminus x_1)$ and $y_2 = x_2 \vee (y \setminus x_2)$. Thus $e = x_1 \vee (y_1 \setminus x_1) \vee z$ and $f = x_2 \vee (y_2 \setminus x_2) \vee z$. Hence x_1 oc $(y_1 \setminus x_1) \vee z$ and x_2 oc $(y_2 \setminus x_2) \vee z$. Since x_1 op x_2, x_1 oc $(y_2 \setminus x_2) \vee z$ and thus $(y_1 \setminus x_1) \vee z$ op $(y_2 \setminus x_2) \vee z$. By Lemma 8 it follows that $(y_1 \setminus x_1)$ op $(y_2 \setminus x_2)$.

<u>Definition 5</u>: $\overline{y} \setminus \overline{x} = \overline{y \setminus x}$.

By the preceding remarks this subtraction is well defined. Note that <u>it is</u> only a partial subtraction operation since it is only defined when $x \le y$.

<u>Lemma 9</u>: $\overline{y} \setminus \overline{x} \le \overline{y}$

<u>Proof</u>: Follows from $y \setminus x \le y$.

<u>Theorem 12</u>: The greatest lower bound of $\overline{x}$ and $\overline{y \setminus x}$ is $\overline{0}$.

<u>Proof</u>: Suppose $\overline{z} \le \overline{x}$ and $\overline{z} \le \overline{y \setminus x}$, then for all $z_1 \varepsilon \overline{z}$ there exists $x_1 \varepsilon \overline{x}$ and $y_2 \setminus x_2 \varepsilon \overline{y \setminus x}$ with $z_1 \le x_1$ and $z_1 \le y_2 \setminus x_2$. Hence $z_1 \le x_1 \wedge (y_2 \setminus x_2)$. Since op respects $\perp$, $x_1 \perp y_2 \setminus x_2$ and thus $x_1 \wedge (y_2 \setminus x_2) = 0$. Therefore $\overline{z} = \overline{0}$.

The corresponding result for upper bounds does not hold, as will be shown later.

<u>Definition 6</u>: If $\overline{x} \perp \overline{y}$ then $\overline{x} \oplus \overline{y} = \overline{x \vee y}$. $\overline{x} \oplus \overline{y}$ is called the orthogonal sum of $\overline{x}$ and $\overline{y}$.

By Theorem 10 this is well defined. As with the subtraction operation the orthogonal sum is only a partial operation.

<u>Lemma 10</u>: If $\overline{x} \perp \overline{y}$, then $\overline{x} \le \overline{x} \oplus \overline{y}$.

<u>Proof</u>: Let $x_1 \varepsilon \overline{x}$. Then there exists $y_1 \varepsilon \overline{y}$ with $x_1 \perp y_1$. Hence $x_1 \vee y_1$ exists and $x_1 \le x_1 \vee y_1$. Therefore $\overline{x}_1 \le \overline{x_1 \vee y_1} = \overline{x} \oplus \overline{y}$.

<u>Lemma 11</u>: $\overline{x} \perp \overline{y}$ implies $(\overline{x} \oplus \overline{y}) \setminus \overline{x} = \overline{y}$.

<u>Proof</u>: From Lemma 10 the subtraction is defined. $(\overline{x} \oplus \overline{y}) \setminus \overline{x} = (\overline{x \vee y}) \setminus \overline{x} = \overline{(x \vee y) \setminus x}$. Since $x \wedge y = 0$, $(x \vee y) \setminus x = y$ and thus $\overline{(x \vee y) \setminus x} = \overline{y}$. Therefore $(\overline{x} \oplus \overline{y}) \setminus \overline{x} = \overline{y}$.

<u>Lemma 12</u>: $\overline{x} \leq \overline{y}$ implies $\overline{x} \oplus (\overline{y} \setminus \overline{x}) = \overline{y}$.

<u>Proof</u>: Let $x \leq y$, then $x \vee (y \setminus x) = y$. Also $x \wedge (y \setminus x) = 0$. Hence the orthogonal sum of $\overline{x}$ and $\overline{(y \setminus x)}$ exists and $\overline{x} \oplus (\overline{y} \setminus \overline{x}) = \overline{y}$.

<u>Theorem 13</u>: If $\overline{x} \perp \overline{y}$ then $\overline{x} \oplus \overline{y}$ is a minimum upper bound for $\overline{x}$ and $\overline{y}$.

<u>Proof</u>: By Lemma 10 $\overline{x} \oplus \overline{y}$ is an upper bound. Suppose $\overline{z}$ is an upper bound with $\overline{z} \leq \overline{x} \oplus \overline{y}$. For all $x_1 \in \overline{x}$ there exists $z_1 \in \overline{z}$ and $y_1 \in \overline{y}$ with $x_1 \leq z_1 \leq x_1 \vee y_1$. Thus by the antitone law of subtraction $(x_1 \vee y_1) \setminus z_1 \leq (x_1 \vee y_1) \setminus x_1 \leq x_1 \vee y_1$ and hence $(x_1 \vee y_1) \setminus z_1 \leq y_1 \leq x_1 \vee y_1$.

$\overline{y} \leq \overline{z}$ implies there exists $z_2 \in \overline{z}$ with $y_1 \leq z_2$. Thus $(x_1 \vee y_1) \setminus z_1 \leq y_1 \leq z_2$. Because $z_1 \perp (x_1 \vee y_1) \setminus z_1$ and z_2 op z_1, it follows from Lemma 4 that $z_2 \perp (x_1 \vee y_1) \setminus z_1$. Hence $[(x_1 \vee y_1) \setminus z_1] \wedge z_2 = (x_1 \vee y_1) \setminus z_1 = 0$. Thus $x_1 \vee y_1 \leq z_1$, which means that $\overline{x_1 \vee y_1} = \overline{x} \oplus \overline{y} \leq \overline{z}$. Therefore $\overline{x} \oplus \overline{y} = \overline{z}$.

In the general case $\overline{x} \oplus \overline{y}$ is not a least upper bound, however.

It is now appropriate to introduce the concept of an associative orthoalgebra is developed by Patricia Frazer Lock when she was a student of Foulis and Randall.

<u>Definition 7</u>:[42] An associative orthoalgebra is a set L with a binary relation $\perp$, a partial binary operation $\oplus$ defined if and only if $x \perp y$, an orthocomplementation $^{\perp}$, and constants 0 and 1 satisfying the following properties:

(a) $x \perp y$ if and only if $y \perp x$ and $x \oplus y = y \oplus x$

(b) $x \perp 0$ and $x \oplus 0 = x$

(c) $x \perp x^{\perp}$ and $x \oplus x^{\perp} = 1$

(d) $x \perp (x^{\perp} \oplus y)$ implies $y = 0$

(e) $x \perp (x \oplus y)$ implies $x = 0$

(f) $x \perp y$ implies $x \perp (x \oplus y)^{\perp}$ and $y^{\perp} = x \oplus (x \oplus y)^{\perp}$ (orthomodular identity).

(g) $x \perp y$ and $z \perp (x \oplus y)$ implies $y \perp z$ and $x \perp (y \oplus z)$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ (Associative Law)

<u>Theorem 13</u>: If S is a DASBAM then the op logic $(\overline{S}, \leq, \perp, \oplus, ^{\perp}, \overline{0}, \overline{1})$ is an associative orthoalgebra.

<u>Proof</u>: It is necessary to verify the axioms.

(a) $\overline{x} \perp \overline{y}$ if and only if $\overline{y} \perp \overline{x}$ due to the symmetry of $\perp$ in S and also $\overline{x} \oplus \overline{y} = \overline{y} \oplus \overline{x}$ by the commutativity of $\vee$ in S.

(b) $\overline{x} \perp \overline{0}$ as $x \perp 0$ for all $x \in S$. Also $\overline{x} \oplus \overline{0} = \overline{x \vee 0} = \overline{x}$.

(c) $\overline{x} \perp \overline{x}^{\perp}$ as $\overline{x}^{\perp} = \overline{e \setminus x}$ and $x \perp e \setminus x$. $\overline{x} \oplus \overline{x}^{\perp} = \overline{x \vee (e \setminus x)} = \overline{e} = \overline{1}$.

(d) Suppose $\overline{x} \perp (\overline{x}^{\perp} \oplus \overline{y})$, then $x \perp (e \setminus x) \vee y$. Hence $y = 0$ and $\overline{y} = \overline{0}$.

(e) Suppose $\overline{x} \perp (\overline{x} \oplus \overline{y})$, then $x \perp x \vee y$ and $x = 0$. Thus $\overline{x} = \overline{0}$.

(f) Let $\overline{x} \perp \overline{y}$. $\overline{x} \leq \overline{x} \oplus \overline{y}$ implies $(\overline{x} \oplus \overline{y})^{\perp} \leq \overline{x}^{\perp}$, thus $\overline{x}^{\perp} \geq (\overline{x} + \overline{y})^{\perp}$. Also $\overline{y}^{\perp} = \overline{x \vee (e \setminus (x \vee y))} = \overline{x} \oplus \overline{(x \vee y)}^{\perp}$ $= \overline{x} \oplus (\overline{x} \oplus \overline{y})^{\perp}$ since S is orthomodular.

(g) Suppose $\overline{x} \perp \overline{y}$ and $\overline{z} \perp (\overline{x} \oplus \overline{y})$. Then $z \perp x \vee y$ and thus $z \perp y$. $x \vee (y \vee z)$ exists and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = 0$, thus $x \perp (y \vee z)$. Hence $\overline{z} \perp \overline{y}$ and $\overline{x} \perp (\overline{z} \oplus \overline{y})$.

Finally $\overline{x} \oplus (\overline{y} \oplus \overline{z}) = \overline{x \vee y \vee z} = (\overline{x} \oplus \overline{y}) \oplus \overline{z}$.

This gives the structure of the op logic of any general DASBAM.

<u>Theorem 14</u>: (P. Lock) If $(L, \perp, \oplus, {}^{\perp}, 0, 1)$ is an associative orthoalgebra then it is the op logic of some DASBAM.[13]

This last theorem will not be proved here, but it will be used later.

The next section will present some examples of op logics arising from DASBAMs presented in Part Four of Chapter III. Many of these satisfy other special conditions which do not hold in general.

PART TWO: Examples of OP Logics

In this part some examples of op logics will be described. These will mostly arise from DASBAMs described in Part Four of Chapter III.

Example 1: Let S be a classical DASBAM (that is, a Boolean algebra). There is only one operation and therefore the relations of oc and op are trivially the complementation and equality, respectively. The op logic is identical to the DASBAM.

Example 2: Let S be a semiclassical DASBAM. The only non-trivial op pairs are among the maximal elements. The structure of $\bar{S}$ is almost the same as that of S except that all of the tests are squeezed together at the top, as illustrated below in Figure 2.

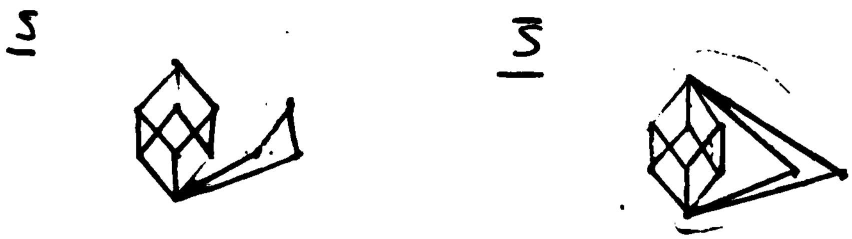


Figure 2

This structure is not Boolean and non-modular since it contains D_5 (circled) as a sublattice. It is orthomodular, though, as was proved in Part One.

Example 3: Let S be the semiclassical DASBAM shown below in Figure 3(a).

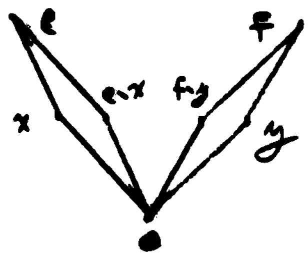


Figure 3(a)

The op logic is shown below in Figure 3(b).

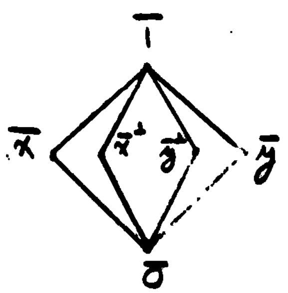


Figure 3(b)

This structure is an orthomodular lattice, specifically OM6.

Example 4: Let S be the free subtraction algebra on two generators. The op logic is shown in Figure 4.

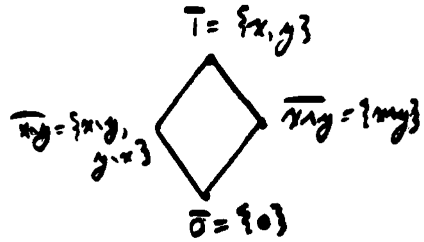


Figure 4

This structure is the Boolean algebra B_4. Whenever the op logic of a DASBAM is a Boolean algebra the DASBAM is called a Boolean DASBAM.

Example 5: Let S be as in Example 4 and form S^+, the dacification of S. The op-logic is shown in Figure 5.

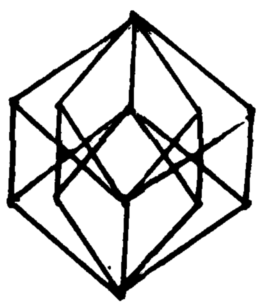


Figure 5

This structure is also an orthomodular lattice, the direct product of OM6 and B_2. The picture in Figure 5 is the same picture that appeared in Scientific American of October 1981 in the article "Quantum Logics" by Hughes. In that article the diagram arose from considering subspaces in $\mathbb{R}^3$, Euclidean 3-space. In the same way this diagram can be interpreted as representing the op-logic of the manual $\{\{x,y,z\}, \{u,v,z\}\}$. These represent orthonormal bases of $\mathbb{R}^3$ by letting $x = [1,0,0]$, $y = [0,1,0]$, $z = [0,0,1]$, $u = [\sqrt{2}/2, \sqrt{2}/2, 0]$, and $v = [-\sqrt{2}/2, \sqrt{2}/2, 0]$. The subspace spanned by x and y is the same as that spanned by u and v.[14]

Example 6: Let S be the Lookout's DASBAM. The partial order diagram of $\overline{S}$ is shown in Figure 6.

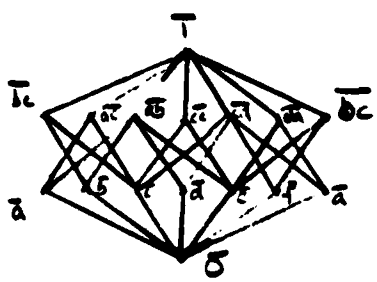


Figure 6

Where $\overline{a} = \{a\}$

$\overline{b} = \{b\}$

$\overline{c} = \{c\}$

$\overline{d} = \{d\}$

$\overline{e} = \{e\}$

$\overline{f} = \{f\}$

$\overline{ab} = \{ab,de\} = \overline{de}$

$\overline{ac} = \{ac\}$

$\overline{bc} = \{bc,ef\} = \overline{ef}$

$\overline{cd} = \{cd,af\}$

$\overline{ce} = \{ce\}$

$\overline{ef} = \{ef,bc\} = \overline{bc}$

$\overline{ae} = \{ae\}$

$\overline{af} = \{af,cd\} = \overline{cd}$

To each equivalence class can be assigned a statement which corresponds to the physical situation. These are listed below:

$\overline{a}$ - contact in sector I

$\overline{b}$ - no contact to port

$\overline{c}$ - contact in sector II

$\overline{d}$ - no contact to starboard

$\overline{e}$ - contact in sector III

$\overline{f}$ - no contact astern

$\overline{ab}$ - no contact in sector II

$\overline{ac}$ - contact in sector I or II

$\overline{bc}$ - no contact in sector I

$\overline{cd}$ - no contact in sector III

$\overline{ce}$ - contact in sector II or III

$\overline{ae}$ - contact in sector I or III

$\overline{1}$ - watch posted

$\overline{0}$ - watch secured

These interpretations make sense in terms of the physical situation. The op relation here preserves physical meaning.

<u>Example 7</u>: Let H be a Hilbert space and $\mathcal{Q}$ to be the Hilbert manual as defined n Example 12 of Chapter III. As explained there each op equivalence class can be identified with a subspace of H. Therefore the op logic is isomorphic to L (H), the lattice of subspaces of H. The order on the logic is the same as on $\mathcal{L}$(H). For any Hilbert manual the op logic is thus an orthomodular lattice.

Take H to be $\mathbb{C}^2$, Complex 2-space. The partial order diagram of the op logic is shown in Figure 7.

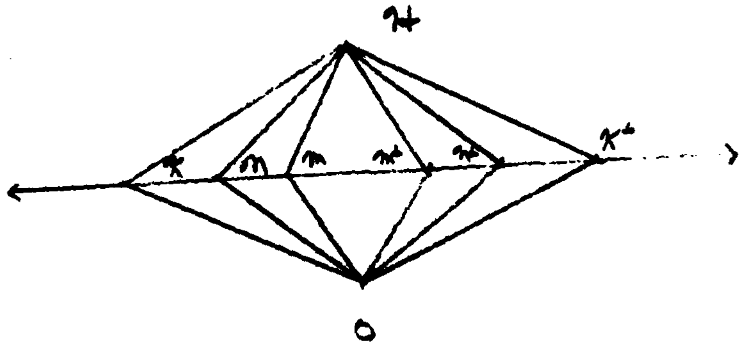


Figure 7

CHAPTER V: RING DASBAMs

The previous discussions of manuals and DASBAMs were motivated by considerations of empirical logic. Many of the examples arose from applications to empirical science. The examples and results in this chapter, on the other hand come from an area which seems to be unrelated to empirical logic. This is the area of abstract algebra and specifically ring theory.

PART ONE: Boolean Rings

Definition 1: A Boolean ring $(R, +, \cdot)$ is a ring satisfying these two identities for all $x \in R$.

BR1	$x^2 = x$	(Idempotent law)
BR2	$x + x = 0$	(Characteristic 2)

From these two seemingly innocuous properties much structure can be derived.

Lemma 1: If R is a Boolean ring then it is commutative, that is $xy = yx$ for all x and y.

Proof: $(x + y)^2 = (x + y)(x + y) = x^2 + yx + xy + y^2 = (x + y) + yx + xy$. But by the idempotent law $(x + y)^2 = x + y$. Thus $yx + xy = 0$.

Hence $yx = -xy = xy$.

Theorem 1: If R is a Boolean ring then it is a subtraction algebra under $x \setminus y = x + xy$.

Proof: As in previous chapters the axioms shall be verified.

(S1) $x \setminus (y \setminus x) = x \setminus (y + yx) = x + xy + x^2 y = x + xy + xy = x.$

(S2) $x \setminus (x \setminus y) = x \setminus (x + xy) = x + x^2 + x^2 y = x + x + xy = xy$ $= yx = y \setminus (y \setminus x).$

(S3) $(z \setminus x) \setminus y = (z + zx) \setminus y = z + zx + zy + zxy$ is symmetric in x and y. Hence their roles can be switched and $(z \setminus x) \setminus y = (z \setminus y) \setminus x.$

This is the quickest way to verify that R is a partially ordered set and is closed under greatest lower bounds.

Lemma 2: $x \leq y$ if and only if $x = xy$.

Proof: Let $x \leq y$, then $x \setminus y = 0 = x + xy$. Thus $xy = -x = x$. Conversely, let $x = xy$. Then $x \setminus y = xy \setminus y = xy + xy^2 = xy + xy = 0$. Hence $x \leq y$.

Lemma 3: For all $x \in R$, $0 \leq x$.

Proof: $0 = 0x$, thus by Lemma 2 $0 \leq x$.

Boolean rings are also closed under least upper bounds.

Theorem 2: If R is a Boolean ring and $x \in R$ and $y \in R$ then $x + y + xy$ is the least upper bound of x and y.

Proof: First it must be established that $x \leq x + y + xy$ and $y \leq x + y + xy$. This follows from $x(x + y + xy) = x^2 + xy + x^2y = x + xy + xy = x$ and likewise $y(x + y + xy) = y$.

Next, suppose $x \leq z$ and $y \leq z$. That is, $xz = x$ and $yz = y$. Then $(x + y + xy)z = xz + yz + xyz = x + y + xy$ and hence $(x + y + xy) \leq z$.

Therefore $x \vee y = x + y + xy$.

Note that the ring is closed under addition and multiplication which means that the expression $x + y + xy$ is defined for all x and y. Hence $x \vee y$ exists for all x and y.

Theorem 3: Any Boolean ring is a distributive lattice with lower bound in which any principal ideal is a Boolean algebra. This structure is called a generalized Boolean algebra.

Proof: This follows from Theorem 1 and Theorem 2, which established that R was a subtraction algebra and that it was closed under least upper bounds.

There is still one more property desired.

Lemma 4: If a Boolean ring has a multiplicative identity 1 it is an upper bound for the lattice.

Proof: For all x, $1 \cdot x = x$ and thus $x \leq 1$.

Theorem 5: Any Boolean ring with identity is a Boolean algebra.

Proof: The lattice generated by the ring is a subtraction algebra with upper bound 1. Therefore, by Theorem I-17, it is a Boolean algebra.

Corollary: The complement of x in a Boolean ring with identity is $1 + x$.

Proof: $x' = 1 \setminus x = 1 + 1 \cdot x = 1 + x$.

These results provide us with the first example of a ring which generates a DASBAM. In this case the DASBAM is classical. The op logic is identical to the DASBAM.

PART TWO: Fields

The next type of ring to be considered is one which has much additional structure, namely a field. A field is a commutative ring with identity in which every non-zero element has a multiplicative inverse.

Theorem 6: If F is a field it is a partially ordered set under $x \leq y$ if and only if $x^2 = xy$.

Proof: (a) $x^2 = x^2$, hence $x \leq x$ (Reflexive)

(b) Let $x \leq y$ and $y \leq x$. Then $x^2 = xy$ and thus $x = 0$ or $x = y$. Likewise $y^2 = yx$ implies $y = x$ or $y = 0$. If $x = 0$ then $y^2 = yx = 0$ and $y = 0$, therefore $y = x$. (anti-symmetric)

(c) Let $x \leq y$ and $y \leq z$. Then $x^2 = xy$ and $y^2 = yz$. Thus either $y = 0$ or $y = z$. If $y = 0$ then $x = 0$ and $xz = 0$. Thus $x \leq z$. If $y = z$ then $x^2 = xz$ and $x + z$. (Transitive)

This partial order is somewhat trivial, as $x^2 = xy$ implies $x = 0$ or $x = y$ in a field. Again, as with Boolean rings the following lemma is true.

Lemma 5: $0 \leq x$ for all $x \in F$.

Proof: $0^2 = 0 = 0 \cdot x$, hence $0 \leq x$.

The partial order diagram below illustrates the general partial order diagram of a field.

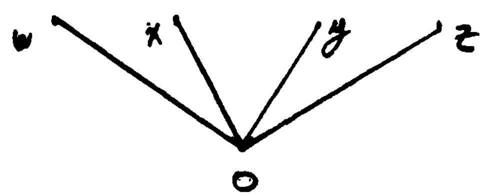


Figure 1

Each non-zero element is an atom and is also maximal. The structure is that of a semi-classical DASBAM where $x \setminus y = x$ if $y \neq x$ and 0 if $y = x$. Every non-zero element is oc to zero and op to each non-zero alement. This structure is called a field DASBAM or field manual.

Example 1: Let $F = Z_5$, the prime field of the integers modulo 5. The partial order diagram is the same as Figure 1.

Example 2: Let $F = GF2^2$, the Galois field of order 4 which is the splitting field of the polynomial $x^3 - 1$ over the prime field Z_2. The partial order diagram is shown in Figure 2, where $\alpha^2 + \alpha + 1 = 0$.[15]

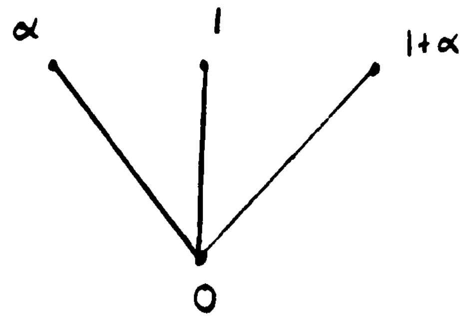


Figure 2

Example 3: Let F = ℝ, the real numbers. The field DASBAM has a non-countably infinite number of atoms and operations. The partial order diagram is shown in Figure 3.

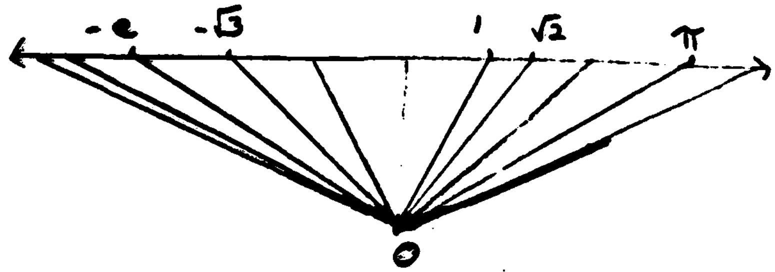


Figure 3

Since the only op equivalence classes are {0} and F∖{0} it follows that the partial order diagram for the op logic of any field DASBAM is that shown in Figure 4.

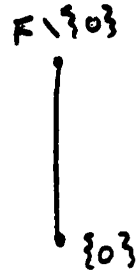


Figure 4

PART THREE: Semi-Simple Rings

<u>Definition 2</u>: A semi-simple ring is a ring R in which $x^n = 0$ for any n implies $x = 0$, that is, R has no non-zero nilpotents.[16]

<u>Theorem 7</u>: (Wardlaw).[17] If R is a finite semi-simple ring it is the direct product of fields.

This result is quite useful when coupled with Theorem III-6, as will be shown by the next Theorem.

<u>Theorem 8</u>: Any finite semi-simple ring forms a DASBAM.

<u>Proof</u>: By Theorem 7 it is the direct product of fields. Since each field is a DASBAM, by Theorem III-6 their direct product is also a DASBAM. Therefore the ring is a DASBAM.

The structure of these ring DASBAMs is rich and interesting. If $R \cong F_1 \times F_2 \times \ldots \times F_n$ is the direct product of n fields then each $x \in R$ can be uniquely expressed as an n-tuple $(x_1,\ldots,x_n)$ where $x_i \in F_i$. The ring operations are the same as the field operations coordinatewise and the DASBAM relations are also the relations from the field DASBAMS expressed coordinatewise. 0 in the ring is $(0,0,0,0\ldots)$ and 1 in the ring is $(1,1,\ldots)$.

<u>Lemma 6</u>: u is invertible if and only if $u_i \neq 0$ for all i.

<u>Proof</u>: Let u be invertible, then there exists u^{-1} such that $uu^{-1} = u^{-1}u = 1 = (1,1,\ldots)$. Then $u_i(u^{-1})_i = 1$ for all i and hence u_i is invertible. Since $x \in F$ is invertible if and only if $x \neq 0$ it follows that $u_i \neq 0$.

Conversely, suppose $u = (u_1,\ldots,u_n)$ where $u_i \neq 0$ for all i. Then $(u_1^{-1}, u_2^{-1},\ldots,u_n^{-1})$ is a multiplicative inverse for u and hence u is invertible.

<u>Theorem 9</u>: x is maximal if and only if x is invertible.

<u>Proof</u>: Let x be maximal and take y to be the n-tuple $(y_1,\ldots,y_n)$ where $y_i = x_i$ when $x_i \neq 0$ and $y_i = 1$ when $x_i = 0$. Clearly $x_i \leq y_i$ and hence $x \leq y$. Thus $x = y$ and $x_i = 0$ for no i. Therefore x is invertible.

Conversely, let x be invertible and suppose that $x \leq y$. Then $x_i \leq y_i$ for all i. $x_i \neq 0$, so $x_i = y_i$ since F_i is a field DASBAM. Therefore $x = y$ and x is maximal.

By this theorem the dominating set of maximal elements is the same as the set of invertible elements.

<u>Theorem 10</u>: x is an atom if and only if x has exactly one non-zero coordinate.

Proof: Let x be an atom and suppose x had two or more non-zero coordinates. x_k and x_ℓ. The element y with $y_i = 0$ when $i \neq \ell$ and $y_\ell = x_\ell$ is not zero, yet $y \leq x$. Hence x is not an atom.

Suppose x has exactly one non-zero coordinate x_k and suppose $y \leq x$. Then $y_i = x_i = 0$ whenever $i \neq k$ and $y_k = x_k$ or 0. Thus $y = x$ or $y = 0$. Therefore x is an atom.

As with direct products of arbitrary DASBAMs the relations of oc and op can be defined coordinatewise.

Theorem 11: x oc y if and only if $x_i = 0$ whenever $y_i \neq 0$ and $x_i \neq 0$ whenever $y_i = 0$.

Proof: $(x \wedge y)_i = x_i \wedge y_i = 0$ for all i and thus $x \wedge y = 0$.

$(x \vee y)_i = x_i \vee y_i$ exists because either x_i or $y_i = 0$. Also $x_i \vee y_i \neq 0$ because either x_i or y_i is non-zero. Therefore $x \vee y$ is invertible and maximal.

Corollary: x oc y if and only if $xy = 0$ and $x + y$ is invertible.

Proof: $(xy)_i = x_i y_i = 0$, thus $xy = 0$. $(x + y)_i = x_i + y_i \neq 0$, thus $x + y$ is invertible.

Theorem 12: x op y if and only if $x_i = 0$ whenever $y_i = 0$ and $x_i \neq 0$ whenever $y_i \neq 0$.

Proof: Let z be defined as such: $z_i = 0$ whenever x_i and y_i are non-zero and $z_i \neq 0$ whenever $x_i = y_i = 0$. Then x oc z and y oc z. Thus x op y.

Definition 3: e is idempotent if and only if $e^2 = e$. $E = \{e \in R | e$ is idempotent$\}$ is the set of all idempotents of R.

This set of idempotents is useful in determining the structure of the op logic.

Theorem 13: $e \in E$ if and only if $e_i = 0$ or 1.

Proof: $e = e^2$ implies $e_i = e_i^2$. Because $e_i \in F_i$, a field, $e_i = 0$ or $e_i = 1$.

Theorem 14: For all $x \in R$ there exists $e_x \in E$ such that x op e_x.

Proof: Take $e_{x_i} = 1$ whenever $x_i \neq 0$ and $e_{x_i} = 0$ whenever $x_i = 0$. e_x is then idempotent and e_x op x.

This shows that $\bar{x} = \bar{e}_x$ for some e_x E. Thus the op-logic is identical to the idempotent structure. The next theorem summarizes this nicely by giving the general structure of the op logic of a finite semi-simple ring DASBAM.

Theorem 15: If R is a finite semi-simple ring DASBAM then $\overline{R}$ is isomorphic to E and is hence Boolean.

Proof: Let $\overline{x} \in \overline{R}$, then there exists $e_x \in E$ such that $\overline{x} = \overline{e_x}$. Also, suppose $\overline{x} \leq \overline{y}$. Then $\overline{e_x} \leq \overline{e_y}$ and $e_x \leq e_y$. Hence the mapping preserves order.

To show E is Boolean note that each idempotent is an n-tuple of zeros and ones. Thus each idempotent can be uniquely identified with the Boolean algebra containing 2^n elements. Therefore the logic and the set of idempotents are Boolean algebras.

There is one more result relating ring structure to the structure of the op-logic.

Theorem 16: Let R be a finite semi-simple ring and $\overline{R} \cong \overline{E}$ the op-logic. There is a one-to-one correspondence between the principal order ideals in the op-logic and the principal ring ideals.

Proof: Let x R be a ring ideal. If $y \in R$ is such that $x_i = 0$ implies $y_i = 0$, the claim is that $y \in x R$.

For take $z_{y_i} = 0$ whenever $y_i = 0$ and $zy_i = x_i^{-1} y_i$ when $y_i \neq 0$. (x_i^{-1} exists because $y_i \neq 0$ and hence $x_i \neq 0$). $x_i z_y = y_i$ for all i, thus $xz_y = y$ and $y \in x R$.

The next claim is that $e_y \leq e_x$. $x_i = 0$ if and only if $e_{x_i} = 0$. Also $x_i = 0$ implies $y_i = 0$ and $ey_i = 0$. Thus $e_{x_i} = 0$ implies $e_{y_i} = 0$ and hence $e_y \leq e_x$. Therefore the principal ideal x R is identified with the principal ideal $I(\overline{e_x})$. Also $e_x \in x R$ as $e_x = x z$ where $z_i = 0$ if $x_i = 0$ and x_i^{-1} if $x_i \neq 0$. Conversely $x \in e_x R$ as $x = e_x x$. Therefore $x R = e_x R$ and $x R = y R$ only if $e_x = e_y$.

Conversely, let $e_x \in E$ and take $I(\overline{e_x})$. This is the set of all elements y where $e_y \leq e_x$. Thus $e_y = e_x e_y$ and $y = e_x e_y y$. Therefore $y \in e_x R$. The order ideal is identical to the ring ideal.

To conclude this section it is best to present some examples.

Example 4: Let $R = Z_6$, the ring of the integers modulo 6. The partial order diagram is shown in Figure 5.

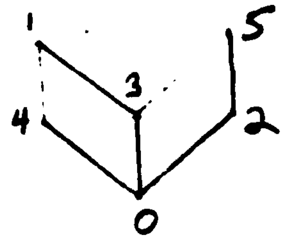


Figure 5

Z_6 is isomorphic to $Z_2 \times Z_3$ as shown below:

0	(0,0)	idempotent
1	(1,1)	idempotent, invertible
2	(0,2)	
3	(1,0)	idempotent
4	(0,1)	idempotent
5	(1,2)	invertible

The number of tests is $2 = \phi(6)$ where ϕ is the Euler phi-function representing the number of elements relatively prime to 6. Z_6 is the product of 2 fields and has $2^2 = 4$ idempotents. As the direct product of field DASBAMs it is a DASBAM, identical to the free DASBAM on two elements.

Example 5: Let $R = Z_{30}$, the ring of the integers modulo 30. Z_{30} is isomorphic to the direct product of the fields Z_2, Z_3, and Z_5. Hence it is a DASBAM. The partial order diagram of Z_{30} and $E(\overline{Z_{30}})$ are given in Figure 6. The maximal elements {1,19,13,29,23,7,11,17}, the invertibles. The atoms are {24,10,18,15,5,20,12}, which have the property that pa = 0 where p = 2,3, or 5 and a is an atom. The idempotents are {1,25,16,21,10, 15,6,0}, those elements in I(1). The principal ring ideal generated by 21 is {0,3,6,9,12,15,18,21,24,27}, which are the same elements contained in $I(\overline{21})$ in the op-logic.

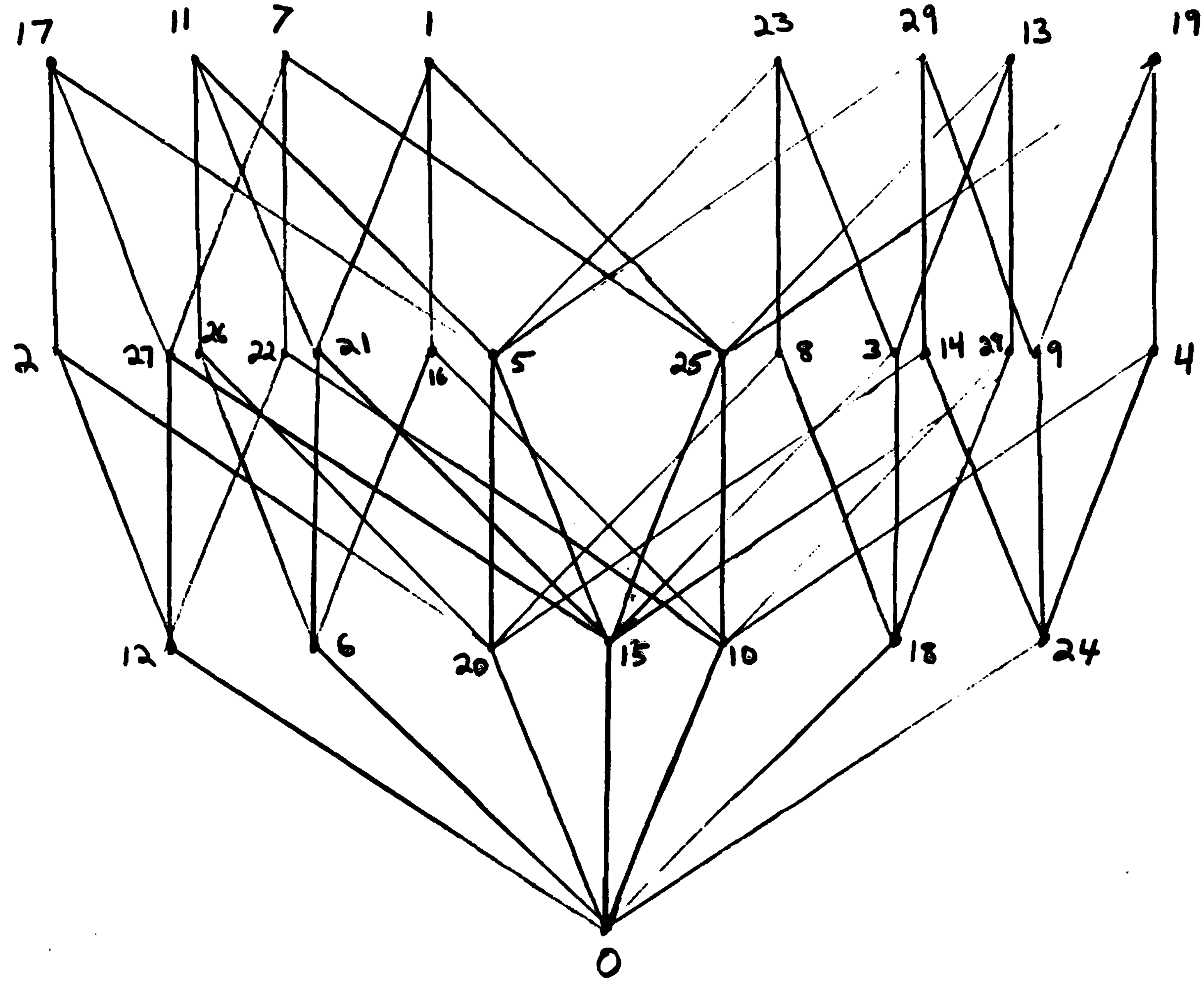

17
11
7
1
23
29
13
19
2
27
26
22
21
16
5
25
8
3
14
28
9
4
12
6
20
15
10
18
24
0

FIGURE 6

PART FOUR: Rings with Involution

The previous sections of this chapter discussed rings which were DASBAMs. In this section rings will be discussed which do not form a DASBAM directly but which generate a DASBAM.

Definition 4: A ring with involution, or *-ring is a ring R with an involution map $* : R \to R$ satisfying the following:

(a) $(x^*)^* = x$

(b) $(x + y)^* = x^* + y^*$

(c) $(xy)^* = y^* x^*$

From the *-ring one can choose a subset $E \leq R$ where $E = \{e \in R | e = e^2 = e^*\}$. These are the idempotent and self-adjoint elements of the ring, also known as the projections of R.

Theorem 17: E is a partially ordered set under $e \leq f$ if and only if $e = ef$.

Proof: (a) $e = e^2$. Thus $e \leq e$ (Reflexive)

(b) Let $e \leq f$ and $f \leq e$. Then $e = ef$ and $f = fe$. Thus $e = e^* = (ef)^* = f^* e^* = fe = f$. (Anti-symmetric)

(c) Let $e \leq f$ and $f \leq g$. Then $e = ef$ and $f = fg$. Hence $eg = (ef)g = e(fg) = ef = e$ and $e \leq g$. (Transitive)

Lemma 7: 0 is a lower bound for E.

Proof: $0 = 0 \cdot 0$. Also $0^* = (0 + 0)^* = 0^* + 0^*$, thus $0^* = 0$. Hence $0 \in E$. Also $0e = 0$ for all $e \in E$, thus $0 \leq e$ for all $e \in E$.

Lemma 8: If the *-ring has a multiplicative identity 1 then 1 is an upper bound for E.

Proof: $1^2 = 1$ and $x^* = (x \cdot 1)^* = 1^* x^*$ for all x, thus $1^* = 1$. Hence $1 \in E$. Finally $e \cdot 1 = e$ for all $e \in E$ and thus $e \leq 1$ for all $e \in E$.

For the rest of this section it will be assumed that R has a multiplicative identity.

Theorem 18: E is orthocomplemented under $e^\perp = 1 - e$.

Proof: (a) $(e^\perp)^\perp = 1 - (1 - e) = e$

(b) Let $f \leq e$ and $f \leq e^\perp$ then $fe = f$ and $f(1-e) = f - fe = f - f = 0 = f$. Thus 0 is the greatest lower bound of e and $e^\perp$.

(c) Let $e \leq g$ and $e^\perp \leq g$, then $eg = e$ and $(1 - e)g = g - eg = g - e$. But $(1 - e)g = 1 - e$, and thus $g = 1$ whence 1 is the least upper bound of e and $e^\perp$.

<u>Definition 5</u>: $e \perp f$ if and only if $ef = 0 = fe$.

<u>Lemma 9</u>: $e \perp f$ if and only if $f \leq 1 - e$

<u>Proof</u>: Let $e \perp f$, then $ef = 0$. Thus $f(1 - e) = f - fe = f - 0 = f$ and $f \leq (1 - e)$.

Let $f \leq (1 - e)$, then $f = f - fe$ and $fe = 0$. Hence $e \perp f$.

<u>Theorem 19</u>: $e \perp f$ implies 0 is the greatest lower bound for e and f and $e + f$ is the least upper bound for e and f.

<u>Proof</u>: Let $e \perp f$ and suppose $g \leq e$ and $g \leq f$. Then $g(ef) = (ge)f = gf = g = 0$, as $ef = 0$. Thus 0 is the greatest lower bound.

Suppose $e \leq g$ and $f \leq g$, then $(e + f)g = eg + fg = e + f$, thus $e + f \leq g$ and $e + f$ is the least upper bound of e and f.

Given the previous results the following theorem, which is the goal of this section shall be established.

<u>Theorem 20</u>: E is an associative orthoalgebra where $\perp$ is as defined in Definition 5. $e^{\perp}$ is $1 - e$. $e \oplus f$ is defined when $e \perp f$ and $e \oplus f = e + f$. Finally 0 and 1 are the constants from the ring.

<u>Proof</u>: The seven axioms of Definition IV-5 need to be verified.

(i) $e \perp f$ implies $ef = 0 = fe$, thus $f \perp e$. Also $e \oplus f = e + f = f + e = f \oplus e$.

(ii) $e \perp 0$ as $e \cdot 0 = e$. Also $e \oplus 0 = e + 0 = e$.

(iii) $e \perp (1 - e)$ as $e(1 - e) = e - e^2 = e - e = 0$. $e \oplus (1 - e) = e + 1 - e = 1$.

(iv) Let $e \perp (f \oplus (1 - e))$, then $ef + e - e^2 = 0$ and thus $ef = 0$. But $f \perp (1 - e)$ and $f - ef = 0$, thus $f = 0$.

(v) Let $e \perp (e \oplus f)$, then $e(e + f) = 0 = e + ef$. But, $e \perp f$ and $ef = 0$, so that $e = 0$.

(vi) Let $e \perp f$, then $e(1 - (e \oplus f)) = e(1 - e - f) = e - e^2 - ef = 0 - 0 = 0$ and $e \perp (e \oplus f)^{\perp}$. Finally $(1 - f) = e + (1 - e - f)$ and thus $f^{\perp} = e \oplus (e \oplus f)^{\perp}$.

(vii) Let $e \perp f$ and $g \perp (e \oplus f)$, then $ge + gf = 0$ and $ef = 0$. Hence $gef + gf^2 = ef$ and $gf = 0$. Thus $g \perp f$. Without loss of generality $g \perp e$. Thus $e(g + f) = eg + ef = 0 + 0 = 0$ and $e \perp (g \oplus f)$. Finally $e \oplus (f \oplus g) = e + (f + g) = (e + f) + g = (e \oplus f) \oplus g$.

<u>Corollary</u>: E generates a DASBAM.

<u>Proof</u>: By Theorem 20, E is an associative ortho-algebra. Thus, by Theorem IV-14 from P. Lock it follows that E is the op logic of some DASBAM.

This DAS?AM may not itself be part of the ring, but is generated from it by the con.truction given in Lock's proof. Basically what is done is maximal orthogonal sets of projections are taken and used to form a manual, from which the event structure is formed to yield a DASBAM.

The next results examine what happens when an additional condition is added to the *-ring. The projection structure satisfies more special properties than a general associative orthoalgebra.

Definition 6: A Baer *-ring is a *-ring satisfying the Baer condition.

For all subsets $S \subseteq R$ there exists $e \in E$ such that the right annihilator of S, i.e. the set $\{x \in \bar{R} \mid sx = 0 \text{ for all } s \in S\}$, is the principal right ideal generated by e, $eR = \{er \mid r \in R\}$. That is, $ser = 0$ for all $s \in S$, $r \in R$.

Theorem 21: If R is a Baer *-ring then E is a complete orthomodular lattice.

Proof: Let $S \subseteq E$ and let $e \in E$ be the annihilator of S, that is $fer = o$ for all $f \in S$, $r \in \bar{R}$. Then $fe1 = 0 = fe$. Hence $f(1 - e) = f - fe = f$ and $f \leq 1 - e$ for all $f \in S$. Let g be such that $f \leq g$ for all $f \in S$. Then $f = fg$ and $f(1-g) = 0$. Hence $1 - g = er$ and $e(1 - g) = e^2r = er = 1 - g$. Thus $1 - g = e - ge$ and $1 - e = g - ge = g(1 - e)$. Therefore $1 - e \leq g$ and $1 - e$ is the least upper bound of S.

To show that the greatest lower bound exists, take $S' = \{1-f \mid f \in S\}$. Let e be the least upper bound for S', then $1 - e$ is the greatest lower bound for S.

Since S is any arbitrary subset of E it follows that E is a complete lattice. The orthomodular law was verified as part (vi) of Theorem 20., hence E is a complete orthomodular lattice.

To illustrate an application of this last theorem the next example is presented.

Example 7: Let H be a Hilbert space and take B(H) to be the set of all bounded linear operators on H. B(H) forms a ring under $(T_1 + T_2)x = T_1x + T_2x$ and $(T_1T_2)x = T_1(T_2x)$. Furthermore, let * be the adjoint on B(H), then B(H) is a *-ring. $P(H) = \{T \in B(H) \mid T = T^2 = T^*\}$ is the set of projections on H and serves the same role as E in R. To show the Baer condition let $\mathcal{S} \leq B(H)$ and form $\bigcap_{S \in \mathcal{S}} \ker S$. Take T to be the projection onto this subspace. Then $ST = 0$ for all $S \in \mathcal{S}$. Furthermore, suppose $SO = 0$ for all $S \in \mathcal{S}$. Then $Ux \in \bigcap_{S \in \mathcal{S}} \ker S$ for all $x \in H$ and thus $U = TU$. Therefore $U \in T\,B(H)$ and the Baer condition is satisfied. The DASBAM generated from P(H) is the same as the Hilbert manual as described in Example III-12.

ENDNOTES

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This paper presents applications of semi-Boolean algebras to empirical logic and ring theory. The development of semi-Boolean algebras from subtraction algebras is shown and the identity of the two is established. Examples of subtraction algebras are given. A weakening of one of the subtraction axioms leads to a structure which is non-distributive but orthomodular. Known as orthosubtraction algebra, this structure is identical to a semi-orthomodular lattice. Since the subspaces of a Hilbert space (and

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thus the projections) form an orthomodular lattice they also form an orthosubtraction algebra. Examples of orthosubtraction algebra applied to Hilbert space are given.

The concept of an manual and how it relates to empirical logic is introduced next. The set of events of a manual is a semi-Boolean algebra. It is atomic and dominated and has relations of operational complemintation and and operational perspectivity defined on it. From these relations the manual condition is defined and the semi-Boolean algebra is a DASBAM. Examples of manuals and DASBAMs are given. In a DASBAM the operational perspectivity relation is an equivalence relation and a quotient structure of equivalence classes modulo this relation can be formed. Known as the op logic, this structure inherits some properties from the DASBAM. It is not a lattice and it is not distributive, however. It does form what is called an associative orthoalgebra. Examples of op logics are given.

The results from semi-Boolean algebras and DASBAMs can be applied to certain types of rings. Boolean rings form classical DASBAMs. Fields form semi-classical DASBAMs in which every atom is a maximal element and vice versa. Semi-simple rings form DASBAMs which are direct products of field DASBAMs. The projections of rings with involution form associative orthoalgebras from which DASBAMs can be generated. In the special case of a Baer *-ring the projections form an orthomodular lattice.